THE ETA INVARIANT OF BERGER SPHERES AND HYPERGEOMETRIC IDENTITIES

MARTIN HABEL AND MANFRED PETER

1. Introduction

In the celebrated index theorem of Atiyah, Patodi and Singer [2] for the Dirac operator on compact Riemannian manifolds with boundary the Eta invariant appears as a global correction term. It is defined in terms of the eigenvalues of the Dirac operator on the boundary as follows: If λ runs through the eigenvalues according to their multiplicities then the Dirichlet series

$$\eta(s) := \sum_{\lambda \neq 0} \frac{\operatorname{sign} \lambda}{|\lambda|^s}, \quad s \in \mathbb{C},$$

converges for sufficiently large $\Re s$ and can be continued analytically to $s = 0$. Its value $\eta(0)$ is called the Eta invariant. If all the other terms in the index formula are known then the Eta invariant can be computed. But often one would like to compute the index of the Dirac operator from the formula and therefore other means to compute $\eta(0)$ are required.

Once the spectrum $\{\lambda\}$ is explicitly known the problem of computing $\eta(0)$ is a purely analytic one. For certain specific examples the spectrum turns out to be parametrized by finitely many discrete variables in a quite elementary way [1], [3], [5]. Thus techniques from Analytic Number Theory might be applicable to compute $\eta(0)$. If there is just one parameter the problem is easy to solve (see, e.g., [11], [12]). The case of several parameters is much harder.

In the present paper the Eta invariant is computed for spheres $\mathcal{S}^{2m+1}$ with Berger metric in which case the eigenvalues depend on two discrete parameters. This is done in two steps:

1. The Dirichlet series $\eta(0)$ is reduced to Dirichlet series associated with certain polynomials and thus $\eta(0)$ is computed in terms of the residues of the latter series at certain half integral values.
2. The residues are expressed in terms of integrals which in the present case can be explicitly evaluated.

Among the known methods for the meromorphic continuation of Dirichlet series associated with polynomials [4], [7], [8], [9], [10], Mahlers method [8] turns out to be best suited for explicit calculations.

Since the Berger metric depends on some positive scaling factor T the Eta invariant is a function of T. The procedure above in fact shows that $\eta(0)$ is a Laurent polynomial in T

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and for every given value of m, it can be computed effectively. Unfortunately, the resulting formula contains rather twisted summations so that not more than this can be seen from it. On the other hand, evaluations with a computer algebra system show the surprising fact that for small values of m the Eta invariant is always of the form $c_m(1-T^2)^{m+1}$. In particular, it is a polynomial in T.

A careful analysis shows that $\eta(0)$ being a polynomial in T is equivalent to certain sums with hypergeometric terms vanishing. We could prove this for all m thus establishing that $\eta(0)$ is indeed a polynomial in T. But we were unable to prove the much stronger conjecture that the Eta invariant is always a monomial in $1-T^2$.

A Pari-GP script for calculating the Eta invariant for arbitrary m can be downloaded from `http://web.mathematik.uni-freiburg.de/mi/zahlen/home/peter`.

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2. Reduction to Dirichlet series associated with polynomials

Let $m \in \mathbb{N}$. The spectrum of the Dirac operator on $\mathcal{S}^{2m+1}$ with Berger metric for the parameter $T > 0$ consists of the following numbers with their respective multiplicities:

$$\begin{aligned}
&\lambda(\nu), (-1)^{m-1}\lambda(\nu) \text{ with multiplicities } \mu(\nu) \quad (\nu \in \mathbb{N}),\\
&\frac{1}{T}\lambda_j^{\pm}(z_1, z_2) \text{ with multiplicities } \frac{\mu_j(z_1, z_2)}{m!j!(m-1-j)!} \quad (z_1, z_2 \in \mathbb{N},\ 0 \le j \le m-1),
\end{aligned}$$

where

$$\begin{aligned}
\lambda(\nu) &:= \frac{1}{T}\Big(\nu + \frac{mT^2 + m - 1}{2}\Big),\\
\mu(\nu) &:= \frac{1}{m!}\prod_{i=0}^{m-1}(\nu + i),\\
\lambda_j^{\pm}(z_1, z_2) &:= \frac{(-1)^j T^2}{2} \pm \sqrt{Q_j(z_1, z_2)},\\
Q_j(z_1, z_2) &:= \Big[(T^2+1)\Big(\frac{m-1}{2} - j\Big) + z_1 - z_2\Big]^2 + 4T^2(z_1 + m - 1 - j)(z_2 + j),\\
\mu_j(z_1, z_2) &:= \frac{z_1 + z_2 + m - 1}{(z_1 + m - 1 - j)(z_2 + j)}\prod_{i=0}^{m-1}(z_1 + i)(z_2 + i).
\end{aligned}$$

For $m = 1$ this result is due to Hitchin [5]. In the general case the spectrum was computed by Bär [3].

For even m the spectrum is symmetric about 0 and thus $\eta(0) = 0$. So from now on we will assume that m is odd. In order to simplify computations we assume that $0 < T < 4\sqrt{m}$. Then

$$Q_j(z_1, z_2) > \Big(\frac{T^2}{2}\Big)^2 \tag{2.1}$$

for $z_1, z_2 \geq 1$ and thus

$$\eta(s) = \eta_1(s) + 2 \sum_{j=0}^{(m-3)/2} \eta_{2,j}(s) + \eta_{2,(m-1)/2}(s), \tag{2.2}$$

where

$$\eta_1(s) := \sum_{\nu \geq 1} \frac{2\mu(\nu)}{\lambda(\nu)^s}, \tag{2.3}$$

$$\eta_{2,j}(s) := \frac{T^s}{m!j!(m-1-j)!} \sum_{z_1,z_2 \geq 1} \mu(z_1,z_2)\big(\lambda_j^+(z_1,z_2)^{-s} - (-\lambda_j^-(z_1,z_2))^{-s}\big) \tag{2.4}$$

for sufficiently large $\Re s$.

For the moment, fix $0 \leq j \leq m-1$ and set $\mu := \mu_j$, $Q := Q_j$, $a := (-1)^j T^2/2$. For $\Re s > 2m+1$, the Binomial series gives

$$D_\pm(s) := \sum_{z_1,z_2 \geq 1} \frac{\mu(z_1,z_2)}{(\sqrt{Q(z_1,z_2)} \pm a)^s} = \sum_{z_1,z_2 \geq 1} \frac{\mu(z_1,z_2)}{Q(z_1,z_2)^{s/2}} \sum_{k \geq 0} \binom{-s}{k} \left(\frac{\pm a}{\sqrt{Q(z_1,z_2)}}\right)^k,$$

where the right hand double series is absolutely convergent. For $\Re s > m + 1/2$, define the Dirichlet series

$$DS[\mu,Q](s) := \sum_{z_1,z_2 \geq 1} \frac{\mu(z_1,z_2)}{Q(z_1,z_2)^s}.$$

Then for $\Re s > 2m+1$, we have

$$D_\pm(s) = \sum_{k \geq 0} \binom{-s}{k} (\pm a)^k DS[\mu,Q]\Big(\frac{s+k}{2}\Big).$$

From (2.1) it follows that there is an $\epsilon > 0$ such that for all $\delta > 0$,

$$DS[\mu,Q](s) \ll_\delta (a^2+\epsilon)^{m-1/2-\Re s} \quad \text{for } \Re s \geq m + \frac{1}{2} + \delta.$$

Thus the series

$$R_\pm(s) := \sum_{k \geq 2m+2} \binom{-s}{k} (\pm a)^k DS[\mu,Q]\Big(\frac{s+k}{2}\Big)$$

is uniformly convergent for $\Re s \geq -1 + 2\delta$, $|s| \leq K$, where $K, \delta > 0$, and consequently it represents a holomorphic function on $\Re s > -1$. Furthermore, we have $R_\pm(0) = 0$. In Proposition 4.1 below it will be shown that $DS[\mu,Q]$ has a meromorphic continuation to $\mathbb{C}$ and only simple poles. Thus

$$D_\pm(s) = \sum_{k=0}^{2m+1} \binom{-s}{k} (\pm a)^k DS[\mu,Q]\Big(\frac{s+k}{2}\Big) + R_\pm(s)$$

has a meromorphic continuation to $\Re s > -1$ and its Laurent expansion at $s = 0$ begins with

$$s^{-1} \cdot 2 \operatorname{res}_{s=0} DS[\mu,Q](s) + s^0 \cdot 2 \sum_{k=1}^{2m+1} \frac{(-1)^k}{k} (\pm a)^k \operatorname{res}_{s=k/2} DS[\mu,Q](s) + \cdots .$$

Together with (2.4) this proves the following proposition.

Proposition 2.1. *For $m \in \mathbb{N}$ odd, $0 \leq j \leq m-1$ and $0 < T < 4\sqrt{m}$, the function $\eta_{2,j}(s)$ has a meromorphic continuation to $\Re s > -1$ and is holomorphic at $s = 0$ with*

$$\eta_{2,j}(0) = \frac{2(-1)^{j+1}T^2}{m!j!(m-1-j)!} \sum_{k=0}^{m} \frac{T^{4k}}{(2k+1)4^k} \operatorname{res}_{s=k+1/2} DS[\mu_j, Q_j](s).$$

3. The one-parameter Dirichlet series

In this section we calculate $\eta_1(0)$.

Proposition 3.1. *Let $\Phi \in \mathbb{C}[X]$ have degree d and $\alpha > 0$. For $\Re s > d+1$, define*

$$D(s) := \sum_{\nu \geq 0} \frac{\Phi(\nu)}{(\nu+\alpha)^s}.$$

Then D has a meromorphic continuation to $\mathbb{C}$ and is holomorphic at 0 with

$$D(0) = -\sum_{l=0}^{d} \frac{B_{l+1}(\alpha)}{(l+1)!} \, \Phi^{(l)}(-\alpha),$$

where B_ν is the ν-th Bernoulli polynomial.

Proof. By linearity we can reduce the general case to the case $\Phi = X^d$. Then

$$D(s) = \sum_{\nu \geq 0} \frac{(\nu+\alpha-\alpha)^d}{(\nu+\alpha)^s} = \sum_{l=0}^{d} \binom{d}{l} (-\alpha)^{d-l} \zeta(s-l,\alpha), \quad \Re s > d+1,$$

where $\zeta(\cdot,\alpha)$ is the Hurwitz zeta function. It has a meromorphic continuation to $\mathbb{C}$ with a single simple pole at $s = 1$; furthermore,

$$\zeta(-k,\alpha) = -\frac{B_{k+1}(\alpha)}{k+1}, \quad k \in \mathbb{N}_0$$

(see [14], Section (13·14)). Now the part about meromorphy of $D(s)$ follows. Furthermore,

$$D(0) = -\sum_{l=0}^{d} \binom{d}{l} (-\alpha)^{d-l} \frac{B_{l+1}(\alpha)}{l+1} = -\sum_{l=0}^{d} \frac{\Phi^{(l)}(-\alpha)}{(l+1)!} B_{l+1}(\alpha).$$

□

Applying this proposition to (2.3) gives

Corollary 3.2. *Let $m \in \mathbb{N}$ be odd and $0 < T < 4\sqrt{m}$. Define $\Phi := \prod_{i=0}^{m-1}(X+i)$. Then*

$$\eta_1(0) = -\frac{2}{m!} \sum_{l=0}^{m} \frac{B_{l+1}((mT^2+m+1)/2)}{(l+1)!} \, \Phi^{(l)}\Big(-\frac{mT^2+m-1}{2}\Big).$$

4. The two-parameter Dirichlet series

Let μ and Q have the same meaning as in Section 2. The residues of $D[\mu, Q]$ are calculated in three steps.

First we get rid of the linear and constant part of Q in the denominator. Set

$$P_2 := (z_1 - z_2)^2 + 4T^2 z_1 z_2, \quad P_1 := Q - P_2.$$

Then it follows that, for $\Re s > m + 1/2$,

$$\begin{aligned} DS[\mu, Q](s) = & \sum_{z_1, z_2 \geq 1 \colon z_1 + z_2 > K} \frac{\mu(z_1, z_2)}{P_2(z_1, z_2)^s} \sum_{h \geq 0} \binom{-s}{h} \left(\frac{P_1}{P_2}(z_1, z_2) \right)^h \\ & + \sum_{z_1, z_2 \geq 1 \colon z_1 + z_2 \leq K} \frac{\mu(z_1, z_2)}{Q(z_1, z_2)^s} = D_1(s) + D_2(s), \end{aligned}$$

where $K \geq 1$ is choosen such that $|P_1/P_2(z_1, z_2)| \leq 1/2$ for $z_1, z_2 \geq 1$, $z_1 + z_2 > K$. Let $H \in \mathbb{N}$. Then

$$\begin{aligned} D_1(s) = & \sum_{0 \leq h \leq H} \binom{-s}{h} \left(DS[\mu P_1^h, P_2](s+h) - \sum_{z_1, z_2 \geq 1 \colon z_1 + z_2 \leq K} \frac{(\mu P_1^h)(z_1, z_2)}{P_2(z_1, z_2)^{s+h}} \right) \\ & + \sum_{z_1, z_2 \geq 1 \colon z_1 + z_2 > K} \frac{\mu(z_1, z_2)}{P_2(z_1, z_2)^s} \sum_{h > H} \binom{-s}{h} \left(\frac{P_1}{P_2}(z_1, z_2) \right)^h. \end{aligned}$$

The last double sum is absolutely and uniformly convergent for s in a compact subset of $\Re s > m - H/2$. In Proposition 4.3 below we will show for arbitrary $P \in \mathbb{C}[z_1, z_2]$ that $DS[P, P_2]$ has a meromorphic continuation to $\mathbb{C}$ with only simple poles. Thus we have the following proposition.

Proposition 4.1. *The Dirichlet series $DS[\mu, Q]$ has a meromorphic continuation to $\mathbb{C}$ with only simple poles. For $k \in \mathbb{N}_0$, we have*

$$\operatorname{res}_{s=k+1/2} DS[\mu, Q](s) = \sum_{0 \leq h \leq 2m} \binom{-k-1/2}{h} \operatorname{res}_{s=h+k+1/2} DS[\mu P_1^h, P_2](s).$$

In the next step we reduce $DS[P, P_2]$ to parameter integrals. Let $\phi \in C^\infty(\mathbb{R}^2)$ with $\phi(x) = 0$ for $\|x\| \leq 1/2$ and $\phi(x) = 1$ for $\|x\| \geq 1$. A twofold application of Euler's sum formula (see, e.g., [13], Chapter I.0, Theorem 4) gives, for $P \in \mathbb{C}[z_1, z_2]$, $d := \deg P$, $\Re s > (d+2)/2$, $L \in \mathbb{N}$,

$$\begin{aligned} DS[P, P_2](s) = & \sum_{z_1, z_2 \geq 1} \frac{\phi P}{P_2^s}(z_1, z_2) \\ = & DI^2[P, P_2](s) - \sum_{1 \leq l \leq L} \frac{(-1)^l B_l}{l!} \int_1^\infty \frac{\partial^{l-1}}{\partial z_2^{l-1}} \frac{P}{P_2^s}(z_1, 0)\, dz_1 \\ & - \sum_{1 \leq l \leq L} \frac{(-1)^l B_l}{l!} \int_1^\infty \frac{\partial^{l-1}}{\partial z_1^{l-1}} \frac{P}{P_2^s}(0, z_2)\, dz_2 + h(s), \end{aligned} \tag{4.1}$$

where

$$DI^2[P,P_2](s) := \int_{\|(z_1,z_2)\|\geq 1} \frac{P}{P_2^s}(z_1,z_2)\,dz_1dz_2, \quad \Re s > \frac{d+2}{2},$$

and $h(s)$ is holomorphic on $\Re s > (d+2-L)/2$. The mollifier ϕ is used to cut out the critical point $(0,0)$ and only appears in the terms collected in $h(s)$.

The next proposition is concerned with the meromorphic continuation of the onedimensional integrals.

Proposition 4.2. *Let $P = z_1^u z_2^v$, $u,v \in \mathbb{N}_0$, and $l \in \mathbb{N}_0$. For $\Re s > (v+1)/2$, define*

$$DI^1_{(z_1)}[P,P_2;l](s) := \int_1^\infty \frac{\partial^l}{\partial z_1^l}\frac{P}{P_2^s}(0,z_2)\,dz_2.$$

This function has a meromorphic continuation to $\mathbb{C}$ with only a simple pole at $s = (u+v+1-l)/2$ and residue

$$\frac{l!}{2}\sum_{k\geq\kappa\geq 0:\,2k-\kappa=l-u}\binom{(1-u-v-l)/2}{k}\binom{k}{\kappa}(4T^2-2)^\kappa.$$

Proof. There is some $\epsilon > 0$ such that for $\Re s > (v+1)/2$, $0 \leq z_1 \leq \epsilon$, we have

$$\begin{aligned}
J(s,z_1) := \int_1^\infty \frac{P}{P_2^s}(z_1,z_2)\,dz_2 &= z_1^u\int_1^\infty z_2^{v-2s}\Big(1+(4T^2-2)\frac{z_1}{z_2}+\frac{z_1^2}{z_2^2}\Big)^{-s}dz_2\\
&= z_1^u\sum_{k\geq 0}\binom{-s}{k}\int_1^\infty z_2^{v-2s}\Big((4T^2-2)\frac{z_1}{z_2}+\frac{z_1^2}{z_2^2}\Big)^k dz_2\\
&= z_1^u\sum_{k\geq 0}\binom{-s}{k}\sum_{0\leq\kappa\leq k}\binom{k}{\kappa}(4T^2-2)^\kappa z_1^{2k-\kappa}\frac{1}{2s-v+2k-\kappa-1}.
\end{aligned}$$

If ϵ is choosen small enough then for every compact $K \in \mathbb{C}$ the double series, after removing finitely many terms, converges uniformly for $s \in K$, $|z_1| < \epsilon$. Thus $J(s,z_1)$ has a meromorphic continuation to $\mathbb{C}\times\{|z_1|<\epsilon\}$ and $I(s) = (\partial^l J)/(\partial z_1^l)(s,0)$ has a meromorphic continuation to $\mathbb{C}$ with only simple poles. They lie at $s = \lambda/2$, $\lambda \in \mathbb{Z}$, $\lambda \leq v+1$, and have

$$\begin{aligned}
\operatorname{res}_{s=\lambda/2} I(s) &= \frac{\partial^l}{\partial z_1^l}\operatorname{res}_{s=\lambda/2} J(s,z_1)\Big|_{z_1=0}\\
&= \frac{\partial^l}{\partial z_1^l}\Big(\frac{1}{2}z_1^u\sum_{k\geq\kappa\geq 0:\,2k-\kappa=v+1-\lambda}\binom{-\lambda/2}{k}\binom{k}{\kappa}(4T^2-2)^\kappa z_1^{2k-\kappa}\Big)\Big|_{z_1=0}\\
&= \frac{l!}{2}\sum_{k\geq\kappa\geq 0:\,2k-\kappa=v+1-\lambda}\binom{-\lambda/2}{k}\binom{k}{\kappa}(4T^2-2)^\kappa
\end{aligned}$$

if $u+v+1-\lambda = l$ and $\operatorname{res}_{s=\lambda/2} I(s) = 0$ otherwise. □

In Proposition 4.4 below it will be shown that $DI^2[P,P_2]$ has a meromorphic continuation to $\mathbb{C}$ with only simple poles. They lie at $s = \lambda/2$, $\lambda \in \mathbb{Z}$, $\lambda \leq d+1$. Thus (4.1) and Proposition 4.2 give

Proposition 4.3. *Let $P \in \mathbb{C}[z_1, z_2]$ with $d := \deg P$. Then $DS[P, P_2]$ has a meromorphic continuation to $\mathbb{C}$ with only simple poles. They lie at $\lambda/2$, $\lambda \in \mathbb{Z}$, $\lambda \le d+1$, and have*

$$\begin{aligned}\operatorname{res}_{s=\lambda/2} DS[P,P_2](s) = &\operatorname{res}_{s=\lambda/2} DI^2[P,P_2](s)\\ &+ \sum_{1\le l\le d+2-\lambda} \frac{(-1)^{l+1}B_l}{l!} \operatorname{res}_{s=\lambda/2} DI^1_{(z_1)}[P,P_2;l-1](s)\\ &+ \sum_{1\le l\le d+2-\lambda} \frac{(-1)^{l+1}B_l}{l!} \operatorname{res}_{s=\lambda/2} DI^1_{(z_2)}[P,P_2;l-1](s).\end{aligned}$$

The last step in the calculation of the residues of $D[\mu, Q]$ is

Proposition 4.4. *Let $P = z_1^g z_2^{G-g}$, $0 \le g \le G$, with G odd. Then $DI^2[P, P_2]$ has a meromorphic continuation to $\mathbb{C}$ with only a simple pole at $s = (G+2)/2$ and residue*

$$\frac{((G-1)/2)!}{4T^{G+1}G!} \sum_{\mu=0}^{(G-1)/2} T^{2\mu} \frac{((G-1)/2-\mu)!(2\mu)!}{\mu!} \binom{g-(G+1)/2+\mu}{2\mu}.$$

Proof. The idea is to introduce suitable polar coordinates so that the curve $P_2(z_1, z_2) = 1$ corresponds to $r = 1$. For $0 < T < 1$, set

$$\begin{aligned}&(z_1,z_2) = r\Big(\frac{\cos\phi}{2T} - \frac{\sin\phi}{2\sqrt{1-T^2}}, \frac{\cos\phi}{2T} + \frac{\sin\phi}{2\sqrt{1-T^2}}\Big) =: r\cdot(z_1(\phi), z_2(\phi)),\\ &r \ge 1,\ |\phi| \le \arctan\Big(\frac{\sqrt{1-T^2}}{T}\Big).\end{aligned}$$

For $T > 1$, set

$$\begin{aligned}&(z_1,z_2) = r\Big(\frac{\cosh\phi}{2T} - \frac{\sinh\phi}{2\sqrt{T^2-1}}, \frac{\cosh\phi}{2T} + \frac{\sinh\phi}{2\sqrt{T^2-1}}\Big),\\ &r \ge 1,\ |\phi| \le \operatorname{arctanh}\Big(\frac{\sqrt{T^2-1}}{T}\Big).\end{aligned}$$

For $T = 1$, set

$$(z_1, z_2) = r\cdot(t, 1-t), \quad r \ge 1,\ 0 \le t \le 1.$$

We will give the proof only in the first case. For $\Re s > (G+2)/2$, we have

$$\begin{aligned}DI^2[P,P_2](s) =& \frac{1}{2T\sqrt{1-T^2}} \int_1^\infty \int_{-\arctan(\sqrt{1-T^2}/T)}^{\arctan(\sqrt{1-T^2}/T)} r^{1+G-2s} z_1(\phi)^g z_2(\phi)^{G-g} d\phi\, dr\\ =& \frac{1}{2T\sqrt{1-T^2}} \frac{1}{2s-G-2} \sum_{\substack{0\le a\le g\\ 0\le b\le G-g}} (-1)^{g-a} \binom{g}{a}\binom{G-g}{b}\\ &\times (2T)^{-a-b}(2\sqrt{1-T^2})^{a+b-G} \int_{-\arctan(\sqrt{1-T^2}/T)}^{\arctan(\sqrt{1-T^2}/T)} (\cos\phi)^{a+b}(\sin\phi)^{G-a-b} d\phi.\end{aligned}$$

If $a+b$ is even the integrand is odd and the integral vanishes. Therefore we set $a+b = G-2\rho$, $a = g-\iota$, and get for the sum

$$(2T)^{-G}\sum_{\rho=0}^{(G-1)/2}\Big(\frac{T^2}{1-T^2}\Big)^{\rho}\sum_{\substack{0\le\iota\le g:\\ 0\le 2\rho-\iota\le G-g}}(-1)^{\iota}\binom{g}{\iota}\binom{G-g}{2\rho-\iota}$$
$$\times\int_{-\arctan(\sqrt{1-T^2}/T)}^{\arctan(\sqrt{1-T^2}/T)}(\cos\phi)^{G-2\rho}(\sin\phi)^{2\rho}d\phi.$$

Substituting $\tau=\sin\phi$ gives for the integral

$$\int_{-\sqrt{1-T^2}}^{\sqrt{1-T^2}}\tau^{2\rho}(1-\tau^2)^{(G-1)/2-\rho}d\tau = 2\sqrt{1-T^2}\sum_{\gamma=0}^{(G-1)/2-\rho}(-1)^{\gamma}\binom{(G-1)/2-\rho}{\gamma}\frac{(1-T^2)^{\rho+\gamma}}{2\rho+2\gamma+1}.$$

Putting everything together gives

$$\begin{aligned}DI^2[P,P_2](s) =&\frac{1}{s-(G+2)/2}\frac{1}{(2T)^{G+1}}\sum_{\rho=0}^{(G-1)/2}\sum_{\gamma=0}^{(G-1)/2-\rho}\binom{(G-1)/2-\rho}{\gamma}\frac{T^{2\rho}(T^2-1)^{\gamma}}{2\rho+2\gamma+1}\\ &\times\sum_{0\le\iota\le g:\,0\le 2\rho-\iota\le G-g}(-1)^{\iota}\binom{g}{\iota}\binom{G-g}{2\rho-\iota}.\end{aligned}$$

Now the residue R at $s=(G+2)/2$ will be simplified. The binomial theorem gives

$$\begin{aligned}R =&\frac{1}{(2T)^{G+1}}\sum_{\mu=0}^{(G-1)/2}(-1)^{\mu}T^{2\mu}\sum_{\rho=0}^{\mu}(-1)^{\rho}\sum_{\substack{0\le\iota\le g:\\ 0\le 2\rho-\iota\le G-g}}(-1)^{\iota}\binom{g}{\iota}\binom{G-g}{2\rho-\iota}\\ &\times\sum_{\gamma=0}^{(G-1)/2-\rho}(-1)^{\gamma}\binom{(G-1)/2-\rho}{\gamma}\binom{\gamma}{\mu-\rho}\frac{1}{2\rho+2\gamma+1}.\end{aligned}\tag{4.2}$$

It is well known (see, e.g., [6], equation $(5\cdot 42)$) that, for $n\in\mathbb{N}_0$, $f\in\mathbb{C}[X]$, $\deg f<n$,

$$\sum_{\nu=0}^{n}(-1)^{\nu}\binom{n}{\nu}f(\nu)=0.\tag{4.3}$$

Now

$$\left(\binom{\gamma}{\mu-\rho}-\binom{-\rho-1/2}{\mu-\rho}\right)\frac{1}{2\rho+2\gamma+1}$$

is a polynomial in γ of degree $\mu-\rho-1<(G-1)/2-\rho$. Therefore replacing $\binom{\gamma}{\mu-\rho}$ by $\binom{\gamma}{\mu-\rho}-\binom{-\rho-1/2}{\mu-\rho}$ in (4.2) makes the innermost sum vanish. Thus

$$\begin{aligned}R =&\frac{1}{(2T)^{G+1}}\sum_{\mu=0}^{(G-1)/2}(-1)^{\mu}T^{2\mu}\sum_{\rho=0}^{\mu}(-1)^{\rho}\sum_{\substack{0\le\iota\le g:\\ 0\le 2\rho-\iota\le G-g}}(-1)^{\iota}\binom{g}{\iota}\binom{G-g}{2\rho-\iota}\binom{-\rho-1/2}{\mu-\rho}\\ &\times\sum_{\gamma=0}^{(G-1)/2-\rho}(-1)^{\gamma}\binom{(G-1)/2-\rho}{\gamma}\frac{1}{2\rho+2\gamma+1}.\end{aligned}$$

Furthermore,

$$\sum_{\nu=0}^{n}(-1)^\nu\binom{n}{\nu}\frac{1}{x+\nu}=\frac{n!}{x(x+1)\cdots(x+n)}$$

(see [6], equation $(5\cdot 41)$). Thus

$$\begin{aligned}R =&\frac{1}{2(2T)^{G+1}}\sum_{\mu=0}^{(G-1)/2}T^{2\mu}\frac{((G-1)/2-\mu)!}{(\mu+1/2)(\mu+3/2)\cdots(G/2)}\sum_{\rho=0}^{\mu}\binom{(G-1)/2-\rho}{(G-1)/2-\mu}\\ &\times\sum_{\substack{0\le\iota\le g:\\ 0\le 2\rho-\iota\le G-g}}(-1)^\iota\binom{g}{\iota}\binom{G-g}{2\rho-\iota}.\end{aligned}\tag{4.4}$$

In Lemma 4.6 below it will be shown that the two inner sums equal

$$2^{2\mu}\binom{g-(G+1)/2+\mu}{2\mu}.$$

Plugging this in (4.4) proves the proposition. □

Lemma 4.5. *For $\alpha\in\mathbb{C}$, $n\in\mathbb{N}_0$, we have*

$$\sum_{\nu=0}^{n}(-1)^\nu\binom{\alpha}{\nu}\binom{n-\alpha-1}{n-\nu}=(-2)^n\binom{\alpha}{n}.$$

Proof. The following elegant argument is due to E. Wirsing. Using the identity $\binom{x}{k}=(-1)^k\binom{k-x-1}{k}$ gives for the left hand side

$$(-1)^n\sum_{\nu=0}^{n}\binom{\alpha}{\nu}\binom{\alpha-\nu}{n-\nu}=(-1)^n\sum_{\nu=0}^{n}\binom{n}{\nu}\binom{\alpha}{n}=(-1)^n\,2^n\binom{\alpha}{n}.$$

□

Lemma 4.6. *Let $G\in\mathbb{N}$ be odd, $0\le g\le G$ and $0\le a\le (G-1)/2$. Then*

$$\sum_{\rho=0}^{(G-1)/2}\binom{(G-1)/2-\rho}{a}\sum_{0\le\iota\le g:\,0\le2\rho-\iota\le G-g}(-1)^\iota\binom{g}{\iota}\binom{G-g}{2\rho-\iota}=(-2)^{G-2a-1}\binom{g-a-1}{G-2a-1}.$$

Proof. Consider the formal power series

$$f_a(X):=\sum_{\nu\ge0}\binom{\nu}{a}X^\nu=X^a(1-X)^{-a-1}.$$

Then the $(G-1)$-st coefficient of

$$g_a(X):=(1-X)^g(1+X)^{G-g}f_a(X^2)$$

is

$$\sum_{\substack{0\le\iota\le g,\,0\le\lambda\le G-g,\,\nu\ge0:\\ \iota+\lambda+2\nu=G-1}}(-1)^\iota\binom{g}{\iota}\binom{G-g}{\lambda}\binom{\nu}{a}$$

$$= \sum_{\substack{0\le\iota\le g,\, 0\le\rho\le(G-1)/2:\\ 0\le 2\rho-\iota\le G-g}} (-1)^\iota \binom{g}{\iota}\binom{G-g}{2\rho-\iota}\binom{(G-1)/2-\rho}{a},$$

which is the left hand side of the identity. On the other hand,

$$g_a(X) = X^{2a}(1-X)^{g-a-1}(1+X)^{G-g-a-1}$$

and thus its $(G-1)$-st coefficient is

$$\sum_{\mu,\nu\ge 0:\, \mu+\nu=G-1-2a} (-1)^\nu \binom{g-a-1}{\nu}\binom{G-g-a-1}{\mu} = (-2)^{G-1-2a}\binom{g-a-1}{G-1-2a}$$

by Lemma 4.5, which is the right hand side of the identity. □

5. The Eta invariant as a function of T

For any given odd $m \in \mathbb{N}$ the Eta invariant can be computed from (2.2), Corollary 3.2 and Propositions 2.1 and 4.1–4.4. In particular, it is clear that $\eta(0)$ is always a Laurent polynomial in T. The next proposition gives more precise information.

Proposition 5.1. *For $m \in \mathbb{N}$ odd, the Eta invariant $\eta(0)$ is always a polynomial in T as long as $0 < T < 4\sqrt{m}$. In particular,*

$$\lim_{T\to 0} \eta(0) = -\frac{2}{m!}\sum_{l=0}^{m} \frac{B_{l+1}((m+1)/2)}{(l+1)!}\ \Phi^{(l)}\Big(-\frac{m-1}{2}\Big).$$

Proof. It is sufficient to prove that

$$T^{2k}\operatorname{res}_{s=k+1/2} DS[\mu_j, Q_j](s)$$

is a polynomial in T for all $0 \le j \le m-1$, $k \ge 0$. Then we have in particular

$$\lim_{T\to 0}\eta(0) = \lim_{T\to 0}\eta_1(0)$$

and the formula follows form Corollary 3.2.

Fix $0 \le j \le m-1$, $k \ge 0$, and let μ, Q, P_1 and P_2 have the same meaning as in Section 4. Propositions 4.1, 4.2 and 4.3 show that it is sufficient to prove that for all $u, v \in \mathbb{N}_0$, $u+v \le 2m-1$, the sum

$$S := T^{2k}\sum_{0\le h\le 2m}\binom{-k-1/2}{h}\operatorname{res}_{s=h+k+1/2} DI^2[z_1^u z_2^v P_1^h, P_2](s)$$

is a polynomial in T. Set $X := m-1+2j$, $Y := 3(m-1)-2j$. Then

$$\begin{aligned} P_1 =& \frac{T^4}{16}(X-Y)^2 - \frac{T^2}{16}(X^2-6XY+Y^2) + \frac{1}{16}(X-Y)^2 \\ &+ \frac{1}{2}z_1(2T^2X+Y-X) + \frac{1}{2}z_2(2T^2Y+X-Y) \end{aligned}$$

and

$$S = T^{2k}\sum_{\substack{\alpha,\beta,\gamma\ge 0:\\ \alpha+\beta+\gamma\le 2m}} \frac{1}{\alpha!\beta!\gamma!}\Big(\frac{T^4}{16}(X-Y)^2 - \frac{T^2}{16}(X^2-6XY+Y^2)\Big)^\alpha$$

$$\times (T^2X)^\beta (T^2Y)^\gamma \sum_{\alpha+\beta+\gamma\le h\le 2m} \binom{-k-1/2}{h} \sum_{\substack{\delta,\epsilon,\zeta\ge 0:\\ \delta+\epsilon+\zeta=h-\alpha-\beta-\gamma}} \frac{h!}{\delta!\epsilon!\zeta!} 2^{-4\delta-\epsilon-\zeta}(-1)^\epsilon$$
$$\times (X-Y)^{2\delta+\epsilon+\zeta} \operatorname{res}_{s=h+k+1/2} DI^2[z_1^{u+\beta+\epsilon} z_2^{v+\gamma+\zeta}, P_2](s).$$

Thus it is sufficient to prove that for all $\alpha, \beta, \gamma \ge 0$, $\alpha, \beta, \gamma \le 2m$, the sum

$$T^{2(k+\alpha+\beta+\gamma)} \sum_{\alpha+\beta+\gamma\le h\le 2m} \binom{-k-1/2}{h} \sum_{\substack{\delta,\epsilon,\zeta\ge 0:\\ \delta+\epsilon+\zeta=h-\alpha-\beta-\gamma}} \frac{h!}{\delta!\epsilon!\zeta!} 2^{-4\delta-\epsilon-\zeta}$$
$$\times (-1)^\epsilon (X-Y)^{2\delta+\epsilon+\zeta} \operatorname{res}_{s=h+k+1/2} DI^2[z_1^{u+\beta+\epsilon} z_2^{v+\gamma+\zeta}, P_2](s)$$

is a polynomial in T. From Proposition 4.4 it follows that the residue does not vanish at most for $h+k+1/2 = (u+v+\beta+\gamma+\epsilon+\zeta+2)/2$. Introducing this new summation condition gives $\delta = -h-2k-\alpha+1+u+v$ and $2\delta+\epsilon+\zeta = -2k-2\alpha-\beta-\gamma+1+u+v =$const. From now on we will use the convention that summands are 0 which contain factorials $\omega!$ or binomial coefficients $\binom{\eta}{\omega}$ with $\omega < 0$. Therefore we must prove that the sum

$$U := T^{2(k+\alpha+\beta+\gamma)} \sum_{\substack{\alpha+\beta+\gamma\le h\le 2m:\\ \delta:=-h-2k-\alpha+1+u+v}} \frac{1}{\delta!}\binom{-k-1/2}{h}$$
$$\times \sum_{\substack{\epsilon,\zeta\ge 0:\\ \epsilon+\zeta=2h+2k-\beta-\gamma-u-v-1}} \frac{h!}{\epsilon!\zeta!} 2^{2h}(-1)^\epsilon \operatorname{res}_{s=h+k+1/2} DI^2[z_1^{u+\beta+\epsilon} z_2^{v+\gamma+\zeta}, P_2](s)$$

is a polynomial in T. From Proposition 4.4 it follows that

$$U = \frac{k! T^{2(\alpha+\beta+\gamma)}}{2(2k)!} \sum_{\substack{\alpha+\beta+\gamma\le h\le 2m\\ (\delta:=-h-2k-\alpha+1+u+v)}} \frac{(-1)^h}{T^{2h}\delta!} \sum_{\mu=0}^{h+k-1} T^{2\mu} \frac{(h+k-1-\mu)!(2\mu)!}{\mu!}$$
$$\times \sum_{\substack{\epsilon,\zeta\ge 0:\\ \epsilon+\zeta=2h+2k-\beta-\gamma-u-v-1}} \frac{(-1)^\epsilon}{\epsilon!\zeta!} \binom{u+\beta+\epsilon-h-k+\mu}{2\mu}. \tag{5.1}$$

For $m, n \in \mathbb{N}_0$, $z \in \mathbb{C}$, we have the identity

$$\sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu}\binom{\nu+z}{m} = (-1)^n \binom{z}{m-n} \tag{5.2}$$

(see, e.g., [6], equation $(5\cdot 24)$). If $2h+2k-\beta-\gamma-u-v-1<0$, the innermost sum in (5.1) is empty. In the opposite case, equation (5.2) gives the value

$$\frac{(-1)^{\beta+\gamma+u+v+1}}{(2h+2k-\beta-\gamma-u-v-1)!} \binom{u+\beta-h-k+\mu}{2\mu-2h-2k+\beta+\gamma+u+v+1}.$$

Thus in any case,

$$U = \frac{(-1)^{\beta+\gamma+u+v+1} k! T^{2(\alpha+\beta+\gamma)}}{2(2k)!} \sum_{\substack{\alpha+\beta+\gamma\le h\le 2m\\ 0\le\mu\le h+k-1}} \frac{(-1)^h T^{2(\mu-h)} (h+k-1-\mu)!}{(-h-2k-\alpha+1+u+v)!}$$

$$\times \frac{(2\mu)!}{\mu!(2h+2k-\beta-\gamma-u-v-1)!}\binom{u+\beta-h-k+\mu}{2\mu-2h-2k+\beta+\gamma+u++v+1}.$$

In order to prove that U is a polynomial in T we must show that, for $p \in \mathbb{N}$, $p > \alpha+\beta+\gamma$, the $(\alpha+\beta+\gamma-p)$-th coefficient of U vanishes, i.e.

$$\binom{u+\beta-k-p}{-2p-2k+\beta+\gamma+u+v+1} \times \sum_{h\in\mathbb{Z}} \frac{(-1)^h(2h-2p)!}{(-h-2k-\alpha+1+u+v)!(h-p)!(2h+2k-\beta-\gamma-u-v-1)!} = 0. \quad (5.3)$$

The various summation conditions are implied by the convention on factorials and binomial coefficients. If $d := -2p-2k+\beta+\gamma+u+v+1 < 0$ then the binomial coefficient vanishes. In the opposite case, we have $n := -p-2k+1-\alpha+u+v > d \geq 0$ since $p > \alpha+\beta+\gamma$. Thus (5.3) is equivalent to

$$\sum_{h\in\mathbb{Z}}(-1)^h\binom{n}{h-p}\binom{2h-2p}{d} = 0.$$

This last identity follows from (4.3) since $d < n$. □

6. A conjecture

Numerical calculations for the first few m showed an astonishing fact: In all cases the Eta invariant $\eta(0)$ not only is a polynomial in T but it is of the form $c_m(1-T^2)^{m+1}$. The constant c_m can easily be deduced from Proposition 5.1. Thus we make the

Conjecture. *For $m \in \mathbb{N}$ odd and $0 < T < 4\sqrt{m}$, we have*

$$\eta(0) = c_m(1-T^2)^{m+1},$$

where

$$c_m := -\frac{2}{m!}\sum_{l=0}^{m}\frac{B_{l+1}((m+1)/2)}{(l+1)!}\ \Phi^{(l)}\Big(-\frac{m-1}{2}\Big).$$

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Mathematisches Institut, Albert-Ludwigs-Universität, Eckerstr. 1, D-79104 Freiburg
E-mail address: manfred.peter@math.uni-freiburg.de