THE PARSEVAL EQUATION FOR ALMOST PERIODIC ARITHMETICAL FUNCTIONS

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Abstract. Almost periodic functions and integrable functions show striking similarities. In this paper the convergence of the Fourier series of almost periodic functions on $\mathbb{Z}$ is investigated and Parseval's equation is generalized. The underlying philosophy is to identify theorems for the space of q-integrable functions $\mathbb{L}^q[0,1]$ which have an analogue in $\mathcal{A}^q$. Theorems relying on more abstract functional analytic properties have the greatest chance to be transferable.

1. Introduction

For $f : \mathbb{Z} \to \mathbb{C}$, $1 \leq q < \infty$, define

$$\|f\|_q := \left(\limsup_{N\to\infty} \frac{1}{2N+1} \sum_{|n|\leq N} |f(n)|^q \right)^{1/q} \in [0,\infty], \quad \|f\|_u := \sup\left\{ |f(n)| \big| n \in \mathbb{Z} \right\}.$$

Let $\mathcal{A}$ be the complex vector space generated by the functions e_α, $\alpha \in \mathbb{R}$, where $e_\alpha(n) := e(\alpha n) := e^{2\pi i \alpha n}$, $n \in \mathbb{Z}$, $\alpha \in \mathbb{R}$. f is called q-almost periodic (uniformly almost periodic) if for every $\epsilon > 0$ there is some $g \in \mathcal{A}$ such that $\|f-g\|_q \leq \epsilon$ ($\|f-g\|_u \leq \epsilon$). The set $\mathcal{A}^q$ of all these functions becomes a Banach space with norm $\|\cdot\|_q$ if functions f_1, f_2 with $\|f_1 - f_2\|_q = 0$ are identified. For the space $\mathcal{A}^u$ of uniformly almost periodic functions this is unnecessary. For $1 \leq p \leq q < \infty$, $\mathcal{A}^u \subseteq \mathcal{A}^q \subseteq \mathcal{A}^p \subseteq \mathcal{A}^1$. For $f \in \mathcal{A}^1$, the mean value

$$M(f) := \lim_{N\to\infty} \frac{1}{2N+1} \sum_{|n|\leq N} f(n)$$

exists. The numbers $\widehat{f}(\alpha) := M(fe_{-\alpha})$, $\alpha \in \mathbb{R}/\mathbb{Z}$, are called the Fourier coefficients of f. For the theory of almost periodic functions on $\mathbb{N}$, see [10]; on $\mathbb{R}$, see [1]; and on general topological groups, see [7].

The main tool in this paper is the Cauchy convolution $f \times g$ of two functions in $\mathcal{A}^1$. In section 2 its functional analytic properties are listed. An immediate conclusion is

Theorem 1.1. *Let $f \in \mathcal{A}^p$, $g \in \mathcal{A}^q$, $1/p + 1/q = 1$ ($p, q \in [1,\infty) \cup \{u\}$) and $\sum_{\alpha\in\mathbb{R}/\mathbb{Z}} |\widehat{f}(\alpha)\widehat{g}(-\alpha)| < \infty$. Then for $n \in \mathbb{Z}$,*

$$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{|m|\leq N} f(m+n)g(m) = \sum_{\alpha\in\mathbb{R}/\mathbb{Z}} \widehat{f}(\alpha)\widehat{g}(-\alpha)e_\alpha(n).$$

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In section 3 projections of almost periodic functions onto subsums of their Fourier series are constructed. This is the key to the following three results. Let $\mathcal{D}$ be the space of all periodic functions on $\mathbb{Z}$, and $\mathcal{D}^q$ its closure in $\mathcal{A}^q$.

Theorem 1.2. *For $f \in \mathcal{D}^q$ $(q \in [1,\infty) \cup \{u\})$,*

$$\lim_{R\to\infty} \sum_{r|R!,\, 1\le a\le r:\, (a,r)=1} \widehat{f}\left(\frac{a}{r}\right) e_{a/r} = f$$

in $\mathcal{D}^q$.

This is a generalization of a result of Hildebrand ([10], Chap. VI, Theorem 5.1). In the case of q-integrable functions $(1 < q < \infty)$ a theorem of M. Riesz shows that much more is true: f is the $\mathbb{L}^q$-limit of its Fourier series when the latter is summed in natural order (see [3], 12.10.1).

Corollary 1.3. *For $f \in \mathcal{D}^p$, $g \in \mathcal{D}^q$, $1/p + 1/q = 1$ $(p, q \in [1,\infty) \cup \{u\})$,*

$$\lim_{R\to\infty} \sum_{r|R!,\, 1\le a\le r:\, (a,r)=1} \widehat{f}\left(\frac{a}{r}\right) \overline{\widehat{g}\left(\frac{a}{r}\right)} = M(f\overline{g}).$$

Again in the case of integrable functions and $1 < p, q < \infty$, much more is true (see [3], 10.5.4).

Theorem 1.4. *Let $f \in \mathcal{A}^p$ $(p \in [1,\infty) \cup \{u\})$ and assume that the elements $0 \neq \alpha \in \mathbb{R}/\mathbb{Z}$ with $\widehat{f}(\alpha) \neq 0$ are linearly independent over $\mathbb{Z}$. Then*

$$f = \sum_{\alpha\in\mathbb{R}/\mathbb{Z}} \widehat{f}(\alpha) e_\alpha$$

in $\mathcal{A}^p$. If $g \in \mathcal{A}^q$, $1/p + 1/q = 1$, then

$$M(f\overline{g}) = \sum_{\alpha\in\mathbb{R}/\mathbb{Z}} \widehat{f}(\alpha)\overline{\widehat{g}(\alpha)},$$

the sum being absolutely convergent.

This result is already known for uniformly almost perodic functions on $\mathbb{R}$ ([1], Chap. I, § 10.1).

There is an analogous notion of almost periodicity for measurable functions $f : [0,\infty) \to \mathbb{C}$. Let

$$\Delta(x) := \sum_{n\le x} d(n) - x(\log x + 2\gamma - 1), \quad x \ge 1,$$

be the remainder in Dirichlet's divisor problem, and

$$F(t) := t^{-1/2}\Delta(t^2), \quad t \ge 1.$$

Combining the results of Heath-Brown [5] with the reasoning in the proof of Theorem 1.4 gives

Theorem 1.5. *Let $K > 2$ such that*

$$\int_0^X |\Delta(x)|^K dx \ll_\epsilon X^{1+K/4+\epsilon} \tag{1.1}$$

for every $\epsilon > 0$ (for example, $K = 28/3$ is allowed). Then for $q \in \mathbb{N}$, $q < K$,

$$M(F^q) = 2(2\pi\sqrt{2})^{-q} \sum_{0<p<q/2} \binom{q}{p} \cos\left(\frac{\pi}{4}(q-2p)\right) S(p,q)$$

if q is odd; if q is even, there is the additional summand

$$(2\pi\sqrt{2})^{-q} \binom{q}{q/2} S(q/2,q).$$

Here

$$S(p,q) := \sum \frac{d(m_1)\cdots d(m_q)}{(m_1\cdots m_q)^{3/4}}, \quad 0 < p \le q/2,$$

where $m_1, \ldots, m_q$ run through all natural numbers such that $\sqrt{m_1} + \cdots + \sqrt{m_p} = \sqrt{m_{p+1}} + \cdots + \sqrt{m_q}$.

The existence of $M(F^q)$ was already proved by Heath-Brown [5]. In cases $q = 3$ and 4, Tsang [11] even got results about the rate of convergence and gave series representations for $M(F^q)$. Theorem 1.5 extends these representations to all cases covered in [5].

The next result is proved by transfering the result of M. Riesz mentioned above to a special type of almost periodic functions.

Theorem 1.6. *Let $F : [0,1] \to \mathbb{C}$ be Riemann integrable, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and define $f(n) := F(\{\alpha n\})$, $n \in \mathbb{Z}$. Let*

$$\widehat{F}(k) := \int_0^1 F(x) e(-kx)\, dx, \quad k \in \mathbb{Z},$$

be the Fourier coefficients of F.

(a) *For $1 < q < \infty$, $f \in \mathcal{A}^q$ and*

$$f = \lim_{K\to\infty} \sum_{|k|\le K} \widehat{F}(k) e_{k\alpha}$$

in $\mathcal{A}^q$.

(b) *If $g \in \mathcal{A}^p$, $1 < p < \infty$, then*

$$M(f\overline{g}) = \lim_{K\to\infty} \sum_{|k|\le K} \widehat{F}(k)\overline{\widehat{g}(k\alpha)}.$$

Note that for $p \ge 2$ the second part can be proved without using the theorem of M. Riesz.

Corollary 1.7. *Let $g \in \mathcal{A}^p$, $1 < p < \infty$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, and $0 \le a \le b \le 1$. Then*

$$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{|n|\le N:\, \{\alpha n\}\in[a,b]} g(n) = (b-a)M(g) + \sum_{k\ne 0} \frac{e(kb)-e(ka)}{2\pi i k} \widehat{g}(k\alpha),$$

where the series is to be summed symmetrically. In particular for $g \in \mathcal{D}^p$,

$$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{|n|\le N:\,\{\alpha n\}\in[a,b]} g(n) = (b-a)M(g).$$

The second part of this corollary can be proved without appealing to M. Riesz' theorem.

The last two theorems give further conditions under which Parseval's equation with the natural order of summation is true. The first condition restricts the growth of the Fourier coefficients of one of the functions and thus allows to apply the Hausdorff-Young inequality. The second condition restricts the space of the permitted almost periodic functions and thus Hildebrand's deep theorem ([10], Chap. 5, Theorem 1.2) can be applied.

Theorem 1.8. *Let $1 \le p \le 2$, $2 \le q < \infty$ or $q = u$, $1/p + 1/q = 1$, $f \in \mathcal{A}^p$, $g \in \mathcal{A}^q$ and $\sum_{\alpha\in\mathbb{R}/\mathbb{Z}} |\widehat{g}(\alpha)|^p < \infty$. Then*

$$M(f\overline{g}) = \sum_{\alpha\in\mathbb{R}/\mathbb{Z}} \widehat{f}(\alpha)\overline{\widehat{g}(\alpha)},$$

where the series is absolutely convergent.

For $r \in \mathbb{N}$, let

$$c_r := \sum_{1\le a\le r:\,(a,r)=1} e_{a/r}$$

be the r-th Ramanujan sum. Let $\mathcal{B}$ be the complex vector space generated by the Ramanujan sums and $\mathcal{B}^q$ its closure in $\mathcal{A}^q$. For $f \in \mathcal{A}^1$,

$$a_r(f) := \frac{1}{\varphi(r)} M(fc_r)$$

is called the r-th Ramanujan coefficient of f.

Theorem 1.9. *Let $f \in \mathcal{B}^p$, $g \in \mathcal{B}^q$, $1/p + 1/q = 1$ ($p, q \in [1,\infty) \cup \{u\}$). Then for $a \in \mathbb{Z}\setminus\{0\}$,*

$$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{|n|\le N} f(n+a)g(n) = \sum_{r\ge 1} a_r(f)a_r(g)c_r(a).$$

The left hand side, as a function of a, is called the correlation function of f and g. For related results, see Schwarz [8], [9].

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2. The Cauchy Convolution

In sections 2, 3 and 6 some results from Fourier theory are cited which are well known in the general theory of compact groups. In the present situation it is also possible to give straightforward elementary proofs for them. Instead of doing this I give some lemmas which might serve as hints how to prove the results elementarily.

For $f, g : \mathbb{Z} \to \mathbb{C}$, $N \in \mathbb{N}$, define

$$C_N(f,g)(n) := \frac{1}{2N+1} \sum_{|m|\le N} f(n-m)g(m), \quad n \in \mathbb{Z}.$$

Theorem 2.1. (a) *For $f \in \mathcal{A}^p$, $g \in \mathcal{A}^1$ $(p \in [1,\infty) \cup \{u\})$,*

$$f \times g := \lim_{N\to\infty} C_N(f,g) \tag{2.1}$$

exists in $\mathcal{A}^p$, and

$$\|f \times g\|_p \leq \|f\|_p \|g\|_1. \tag{2.2}$$

(b) *For $f \in \mathcal{A}^p$, $g \in \mathcal{A}^q$, $1/p + 1/q = 1$ $(p, q \in [1,\infty) \cup \{u\})$, $f \times g$ can be chosen to lie in $\mathcal{A}^u$, such that*

$$f \times g(n) = \lim_{N\to\infty} C_N(f,g)(n),\ n \in \mathbb{Z}, \quad \|f \times g\|_u \leq \|f\|_p \|g\|_q.$$

(c) *For $f, g \in \mathcal{A}^1$, $f \times g = g \times f$ in $\mathcal{A}^1$.*

(d) *For $f, g \in \mathcal{A}^1$, $\widehat{f \times g} = \widehat{f} \cdot \widehat{g}$.*

The next theorem gives an important approximation property of the convolution.

Theorem 2.2. *For every $f \in \mathcal{A}^p$ $(p \in [1,\infty) \cup \{u\})$ and $\epsilon > 0$ there is some $g \in \mathcal{A}^u$ such that*

$$g \geq 0,\ M(g) = 1,\ \|f - f \times g\|_p \leq \epsilon.$$

In the case of $\mathbb{L}^p[0,1]$-spaces g can be choosen independently of f in the following sense: If $(g_n)_{n\geq 1}$ is a Dirac sequence then $\|f - f \times g_n\|_p \to 0$ as $n \to \infty$ for every p-integrable f. In the case of $\mathcal{A}^p$ it can be shown that this stronger version is false.

Corollary 2.3. *For every $f \in \mathcal{A}^p$ $(p \in [1,\infty) \cup \{u\})$ and $\epsilon > 0$ there is some $g \in \mathcal{A}$ with $\|f - g\|_p \leq \epsilon$ and the property: If $\widehat{f}(\alpha) = 0$ for some $\alpha \in \mathbb{R}/\mathbb{Z}$, then $\widehat{g}(\alpha) = 0$.*

Corollary 2.4. *Let $f \in \mathcal{A}^p$ $(p \in [1,\infty) \cup \{u\})$ and $\widehat{f} = 0$. Then $f = 0$ in $\mathcal{A}^p$.*

Proof of Theorem 1.1.

$$h := \sum_{\alpha\in\mathbb{R}/\mathbb{Z}} \widehat{f}(\alpha)\widehat{g}(-\alpha)e_\alpha \in \mathcal{A}^u$$

since the series is absolutely and uniformly convergent. Define $g^*(n) := g(-n)$, $n \in \mathbb{Z}$. By Theorem 2.1(d),

$$\widehat{f \times g^*}(\alpha) = \widehat{f}(\alpha)\widehat{g^*}(\alpha) = \widehat{f}(\alpha)\widehat{g}(-\alpha) = \widehat{h}(\alpha), \quad \alpha \in \mathbb{R}/\mathbb{Z}.$$

By Theorem 2.1(b), we can choose $f \times g^*$ in $\mathcal{A}^u$ with

$$\lim_{N\to\infty} C_N(f, g^*)(n) = f \times g^*(n), \quad n \in \mathbb{Z}.$$

By Corollary 2.4, $f \times g^* = h$. Thus for $n \in \mathbb{Z}$,

$$\sum_{\alpha\in\mathbb{R}/\mathbb{Z}} \widehat{f}(\alpha)\widehat{g}(-\alpha)e_\alpha(n) = h(n) = f \times g^*(n) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{|m|\leq N} f(n-m)g(-m).$$

□

3. A projector

Let $L \subseteq \mathbb{R}$ be a finitely generated $\mathbb{Z}$-module with $\mathbb{Z} \subseteq L$. Let $U := L \cap \mathbb{Q}$. L is finitely generated and torsion free. Thus U has the same properties. Any two elements in U are $\mathbb{Z}$-linearly dependent. Therefore U is cyclic with some generator a/t, $a, t \in \mathbb{N}$, $(a,t) = 1$. Since $1 \in U$, there is some $b \in \mathbb{Z}$ with $ba/t = 1$. Therefore $a|ab = t$ and $a = 1$. L/U is finitely generated and torsion free. Consequently there are $\beta_1, \ldots, \beta_r \in L$ with

$$L/U = \bigoplus_{\rho=1}^{r} \mathbb{Z} \cdot (\beta_\rho \bmod U)$$

and thus

$$L = \bigoplus_{\rho=1}^{r} \mathbb{Z} \cdot \beta_\rho \oplus \mathbb{Z} \cdot \frac{1}{t}.$$

For $N \in \mathbb{N}$, define

$$K_N := \frac{1}{(N+1)^r} \sum_{0 \le K_1, \ldots, K_r \le N} \quad \sum_{\substack{|k_1| \le K_1, \ldots, |k_r| \le K_r,\, 0 \le \tau \le t-1: \\ \alpha := k_1\beta_1 + \cdots + k_r\beta_r + \tau/t}} e_\alpha \in \mathcal{A}.$$

Lemma 3.1. $K_N \ge 0,\ M(K_N) = 1,\ \|K_N\|_1 = 1$.

For $f \in \mathcal{A}$, define $T_L f := \sum_{\alpha \in L/\mathbb{Z}} \widehat{f}(\alpha) e_\alpha$.

Lemma 3.2. *For $f \in \mathcal{A}$, $\lim_{N\to\infty} C(f, K_N) = T_L f$ in $\mathcal{A}^u$.*

Lemma 3.3. *For $p \in [1, \infty) \cup \{u\}$ and $f \in \mathcal{A}$, $\|T_L f\|_p \le \|f\|_p$.*

$T_L f$ is the projection of f onto a subsum of its Fourier series. The lemma above shows in particular that this projection is Lipschitz continuous.

Theorem 3.4. *Let $L \subseteq \mathbb{R}$ be a finitely generated $\mathbb{Z}$-module with $\mathbb{Z} \subseteq L$ and $p \in [1, \infty) \cup \{u\}$. There is a bounded linear operator $T_L^{(p)} : \mathcal{A}^p \to \mathcal{A}^p$ with the properties:*

(a) $\|T_L^{(p)}\| \le 1$

(b) *If $f \in \mathcal{A}^p$ then $\widehat{T_L^{(p)} f}(\alpha) = \widehat{f}(\alpha)$ for $\alpha \in L/\mathbb{Z}$ and $\widehat{T_L^{(p)} f}(\alpha) = 0$ otherwise.*

Proof of Theorem 1.2. For $N \in \mathbb{N}$, define

$$L_N := \left\{ \frac{a}{r} \,\middle|\, a \in \mathbb{Z},\, r \in \mathbb{N},\, (a,r) = 1,\, r|N \right\}.$$

L_N is a $\mathbb{Z}$-module in $\mathbb{R}$ with $\mathbb{Z} \subseteq L_N$. For $g \in \mathcal{D}^q$, let

$$g_N := \sum_{r|N,\, 1 \le a \le r:\, (a,r)=1} \widehat{g}\left(\frac{a}{r}\right) e_{a/r} \in \mathcal{D}.$$

If $\alpha \in \mathbb{R}/\mathbb{Z}$, Theorem 3.4(b) gives $\widehat{T_{L_N}^{(q)} g}(\alpha) = \widehat{g}(\alpha)$ for $\alpha \equiv a/r \bmod 1$, $r|N$, $1 \le a \le r$, $(a,r) = 1$, and $\widehat{T_{L_N}^{(q)} g}(\alpha) = 0$ otherwise. Thus $\widehat{T_{L_N}^{(q)} g} = \widehat{g_N}$ and $T_{L_N}^{(q)} g = g_N$ in $\mathcal{A}^q$ by Corollary 2.4. For $f \in \mathcal{D}^q$, $\epsilon > 0$, choose $g \in \mathcal{D}$ with $\|f - g\|_q \le \epsilon$. Choose $R_0 \in \mathbb{N}$ with

$$\left\{ \alpha \in \mathbb{R}/\mathbb{Z} \,\middle|\, \widehat{g}(\alpha) \ne 0 \right\} \subseteq L_{R_0!}/\mathbb{Z}.$$

Then for $R \geq R_0$, $g_{R!} = g$, and by Theorem 3.4(a)

$$\|f - f_{R!}\|_q \leq \|f - g\|_q + \|(g-f)_{R!}\|_q \leq \epsilon + \|T^{(q)}_{L_{R!}}(g-f)\|_q \leq 2\epsilon.$$

□

Proof of Corollary 1.3. Hölder's inequality gives for $R \in \mathbb{N}$

$$\begin{aligned}\left| M(f\overline{g}) - \sum_{r|R!,\,1\leq a\leq r:\,(a,r)=1} \widehat{f}\left(\frac{a}{r}\right)\overline{\widehat{g}\left(\frac{a}{r}\right)}\right| &= \left| M\left(\left(f - \sum_{r|R!,\,1\leq a\leq r:\,(a,r)=1} \widehat{f}\left(\frac{a}{r}\right) e_{a/r}\right)\overline{g}\right)\right| \\ &\leq \left\| f - \sum_{r|R!,\,1\leq a\leq r:\,(a,r)=1} \widehat{f}\left(\frac{a}{r}\right) e_{a/r}\right\|_p \|g\|_q.\end{aligned}$$

By Theorem 1.2, the right hand side converges to 0 as $R \to \infty$. □

Proof of Theorem 1.4. Let

$$A := \left\{0 \neq \alpha \in \mathbb{R}/\mathbb{Z} \,\middle|\, \widehat{f}(\alpha) \neq 0\right\} = \bigcup_{j\geq 1} A_j,$$

where $A_1 \subseteq A_2 \subseteq \cdots$ is an increasing sequence of finite sets. Let L_j be the $\mathbb{Z}$-module generated by A_j and 1. For $g \in \mathcal{A}^p$ with $\left\{0 \neq \alpha \in \mathbb{R}/\mathbb{Z} \,\middle|\, \widehat{g}(\alpha) \neq 0\right\} \subseteq A$, define

$$g_j := \sum_{\alpha \in A_j \cup \{0\}} \widehat{g}(\alpha) e_\alpha \in \mathcal{A}.$$

If $\alpha \notin L_j/\mathbb{Z}$, $\widehat{T^{(p)}_{L_j} g}(\alpha) = 0 = \widehat{g_j}(\alpha)$. The same holds for $\alpha \in L_j/\mathbb{Z}$, $\widehat{g}(\alpha) = 0$. Now let $\alpha \in L_j/\mathbb{Z}$, $\widehat{g}(\alpha) \neq 0$. Then $\alpha = 0 \bmod 1$ or $\alpha \in A$, and $\alpha = \sum_{\beta \in A_j} c_\beta \beta$, $c_\beta \in \mathbb{Z}$ for $\beta \in A_j$. Since the elements of A are $\mathbb{Z}$-linearly independent, $\alpha \in A_j$ in case $\alpha \neq 0 \bmod 1$. Thus $\widehat{T^{(p)}_{L_j} g}(\alpha) = \widehat{g}(\alpha) = \widehat{g_j}(\alpha)$. If follows that $\widehat{T^{(p)}_{L_j} g} = \widehat{g_j}$ and $T^{(p)}_{L_j} g = g_j$ in $\mathcal{A}^p$.

Now let $\epsilon > 0$. By Corollary 2.3 there is some $g \in \mathcal{A}$ with

$$\|f - g\|_p \leq \epsilon, \quad B := \{0 \neq \alpha \in \mathbb{R}/\mathbb{Z} \,|\, \widehat{g}(\alpha) \neq 0\} \subseteq A.$$

Choose $j_0 \in \mathbb{N}$ with $B \subseteq A_{j_0}$. For $j \geq j_0$, $g_j = g$ and thus

$$\|f - f_j\|_p \leq \|f - g\|_p + \|(g-f)_j\|_p \leq \epsilon + \|T^{(p)}_{L_j}(g-f)\|_p \leq 2\epsilon$$

by Theorem 3.4(a). This proves the first part of Theorem 1.4. The second part is proved just as Corollary 1.3. □

4. Moments of the remainder in the divisor problem

For a measurable function $f : \mathbb{R} \to \mathbb{C}$, the p-norm ($1 \leq p < \infty$) is defined by

$$\|f\|_p := \left(\limsup_{X\to\infty} \frac{1}{2X} \int_{-X}^{X} |f(x)|^p dx\right)^{1/p}$$

and the u-norm by

$$\|f\|_u := \sup\left\{|f(x)| \,\middle|\, x \in \mathbb{R}\right\}.$$

The definition of the space $\mathcal{A}^p_{\mathbb{R}}$ of p-almost periodic functions on $\mathbb{R}$ ($p \in [1,\infty) \cup \{u\}$) is completely analogous to that of $\mathcal{A}^p$ if $\mathcal{A}$ is replaced by $\mathcal{A}_{\mathbb{R}}$ which contains all trigonometric polynomials defined on $\mathbb{R}$. The theorems and proofs of sections 2 and 3 can now be translated verbally into this new situation. All we have to do is to replace sums by the appropriate integrals. In particular, the analogue of Theorem 3.4 is

Theorem 4.1. *Let $L \subseteq \mathbb{R}$ be a finitely generated $\mathbb{Z}$-module and $p \in [1,\infty) \cup \{u\}$. There is a bounded linear operator $T_L^{(p)} : \mathcal{A}_{\mathbb{R}}^p \to \mathcal{A}_{\mathbb{R}}^p$ with the properties:*

(a) $\|T_L^{(p)}\| \leq 1$

(b) *If $f \in \mathcal{A}_{\mathbb{R}}^p$ then $\widehat{T_L^{(p)}f}(\alpha) = \widehat{f}(\alpha)$ for $\alpha \in L$ and $\widehat{T_L^{(p)}f}(\alpha) = 0$ otherwise.*

Let $\mathcal{A}_{[0,\infty)}^p$ be the space of p-almost periodic functions on $[0,\infty)$ which is defined similarly.

Proposition 4.2. *For $p \in [1,\infty) \cup \{u\}$ the restriction*

$$\iota : \mathcal{A}_{\mathbb{R}}^p \to \mathcal{A}_{[0,\infty)}^p, \quad f \mapsto f \restriction [0,\infty)$$

is an isometric isomorphism. For $f \in \mathcal{A}_{\mathbb{R}}^p$, $\widehat{f} = \widehat{\iota(f)}$.

Proof. (a) Let $f \in \mathcal{A}_{\mathbb{R}}$. It is shown that $\|f\|_{p,\mathbb{R}} = \|f\|_{p,[0,\infty)}$. First let $1 \leq p < \infty$ and $\epsilon > 0$. Approximate $z \mapsto |z|^p$ on $\big\{z \in \mathbb{C} \,\big|\, |z| \leq \|f\|_{u,\mathbb{R}}\big\}$ by a polynomial $P(\Re z, \Im z)$ with error $\leq \epsilon$. Let $g := P(\Re f, \Im f)$. Then $\big||f(x)|^p - g(x)\big| \leq \epsilon$ for $x \in \mathbb{R}$ and therefore

$$\begin{aligned} \big|\|f\|_{p,\mathbb{R}}^p - M_{\mathbb{R}}(g)\big| &\leq M_{\mathbb{R}}\big(\big||f|^p - g\big|\big) \leq \epsilon, \\ \big|\|f\|_{p,[0,\infty)}^p - M_{[0,\infty)}(g)\big| &\leq M_{[0,\infty)}\big(\big||f|^p - g\big|\big) \leq \epsilon. \end{aligned}$$

On the other hand, $g \in \mathcal{A}_{\mathbb{R}}$ and therefore $M_{\mathbb{R}}(g) = M_{[0,\infty)}(g)$. This gives $\big|\|f\|_{p,\mathbb{R}}^p - \|f\|_{p,[0,\infty)}^p\big| \leq 2\epsilon$ for all $\epsilon > 0$ and hence the result.

Now let $p = u$. Let $A := \{\alpha \in \mathbb{R} \,|\, \widehat{f}(\alpha) \neq 0\}$. Since $n \mapsto (e_\alpha(n))_{\alpha \in A}$, is a bounded sequence in $\mathbb{C}^{|A|}$ there is a sequence $1 \leq n_1 < n_2 < \cdots$ in $\mathbb{N}$ with $n_{k+1} - n_k \to \infty$ and $(e_\alpha(n_k))_{\alpha \in A} \to z \in \mathbb{C}^{|A|}$ as $k \to \infty$. Then $e_\alpha(n_{k+1} - n_k) = e_\alpha(n_{k+1})/e_\alpha(n_k) \to 1$ as $k \to \infty$. Let $x \in \mathbb{Z}$. Then for k sufficiently large,

$$\begin{aligned} |f(x)| &\leq |f(x) - f(x + n_{k+1} - n_k)| + |f(x + n_{k+1} - n_k)| \\ &\leq \sum_{\alpha \in A} |\widehat{f}(\alpha)| \cdot |1 - e_\alpha(n_{k+1} - n_k)| + \|f\|_{u,[0,\infty)}. \end{aligned}$$

As $k \to \infty$, $|f(x)| \leq \|f\|_{u,[0,\infty)}$ for $x \in \mathbb{Z}$, and hence the result.

(b) Let $f \in \mathcal{A}_{\mathbb{R}}^p$. Choose a sequence $(g_j)_{j\geq 1}$ in $\mathcal{A}_{\mathbb{R}}$ with $\|f - g_j\|_{p,\mathbb{R}} \to 0$ as $j \to \infty$. Then $\|f\|_{p,[0,\infty)} = \lim_{j\to\infty} \|g_j\|_{p,[0,\infty)} = \lim_{j\to\infty} \|g_j\|_{p,\mathbb{R}} = \|f\|_{p,\mathbb{R}}$. Thus ι is isometric and in particular injective.

Let $f \in \mathcal{A}_{[0,\infty)}^p$. Choose a sequence $(g_j)_{j\geq 1}$ in $\mathcal{A}_{\mathbb{R}}$ with $\|f - g_j\|_{p,[0,\infty)} \to 0$ as $j \to \infty$. Then $(g_j)_{j\geq 1}$ is a Cauchy sequence in $\mathcal{A}_{[0,\infty)}^p$ and by (a) also in $\mathcal{A}_{\mathbb{R}}^p$. Let $f^* := \lim_{j\to\infty} g_j$ in $\mathcal{A}_{\mathbb{R}}^p$. Then in particular $\|f^* - g_j\|_{p,[0,\infty)} = \|f^* - g_j\|_{p,\mathbb{R}} \to 0$ as $j \to \infty$ and therefore $\iota(f^*) = \lim_{j\to\infty} g_j = f$ in $\mathcal{A}_{[0,\infty)}^p$. Thus ι is surjective.

(c) Let $f \in \mathcal{A}_{\mathbb{R}}^p$, $\alpha \in \mathbb{R}$. Choose a sequence in $\mathcal{A}_{\mathbb{R}}$ with $\|f - g_j\|_{p,\mathbb{R}} \to 0$ as $j \to \infty$. Then $\|f - g_j\|_{p,[0,\infty)} \to 0$ and therefore

$$\widehat{f}(\alpha) = \lim_{j\to\infty} M_{\mathbb{R}}(g_j e_{-\alpha}), \quad \widehat{\iota(f)}(\alpha) = \lim_{j\to\infty} M_{[0,\infty)}(g_j e_{-\alpha}).$$

Since $g_j e_{-\alpha} \in \mathcal{A}_{\mathbb{R}}$, $M_{\mathbb{R}}(g_j e_{-\alpha}) = M_{[0,\infty)}(g_j e_{-\alpha})$. Hence $\widehat{f}(\alpha) = \widehat{\iota(f)}(\alpha)$. $\square$

This proposition shows that Theorem 4.1 holds verbally also for functions in $\mathcal{A}^p_{[0,\infty)}$, $p \in [1,\infty) \cup \{u\}$.

Proof of Theorem 1.5. Heath-Brown ([5], Section 5, Equation (5.2)) proved that $F \in \mathcal{A}^2_{[0,\infty)}$ and

$$\widehat{F}(\pm 2\sqrt{n}) = \frac{d(n)}{2\pi\sqrt{2}n^{3/4}} e^{\mp i\pi/4}, \quad n \in \mathbb{N},$$

and $\widehat{F}(\alpha) = 0$ otherwise. He also showed ([5], Lemma 4) that if (1.1) holds for some $K > 2$ and arbitrary $\epsilon > 0$ then $\|F\|_q < \infty$ for all $1 \le q < K$. A standard argument then shows that $F \in \mathcal{A}^q_{[0,\infty)}$ for $1 \le q < K$. According to Besicovitch [2], the square roots of positive squarefree integers are linearly independent over $\mathbb{Q}$. For $N \ge 1$, define

$$L_N := \bigoplus_{n \le N:\, \mu(n)^2 = 1} \mathbb{Z} \cdot 2\sqrt{n},$$

$$F_N := \sum_{n \le N:\, \mu(n)^2=1} \sum_{r \in \mathbb{Z}\setminus\{0\}} \frac{d(nr^2)}{2\pi\sqrt{2}n^{3/4}|r|^{3/2}} e^{-\mathrm{sign}(r) i\pi/4} e_{2r\sqrt{n}}. \tag{4.1}$$

Since the series is absolutely convergent, $F_N \in \mathcal{A}^u_{[0,\infty)}$. If $\alpha \in \mathbb{R} \setminus L_N$, Theorem 4.1 gives $\widehat{T^{(q)}_{L_N} F}(\alpha) = 0 = \widehat{F_N}(\alpha)$. The same holds for $\alpha \in L_N$ with $\widehat{F}(\alpha) = 0$. If $\alpha \in L_N$ with $\widehat{F}(\alpha) \neq 0$, then $\alpha = 2r\sqrt{n}$, $r \in \mathbb{Z}\setminus\{0\}$, $n \in \mathbb{N}$, $\mu(n)^2 = 1$, and $\widehat{T^{(q)}_{L_N} F}(\alpha) = \widehat{F}(\alpha) = \widehat{F_N}(\alpha)$. Thus $\widehat{T^{(q)}_{L_N} F} = \widehat{F_N}$ and $T^{(q)}_{L_N} F = F_N$ in $\mathcal{A}^q_{[0,\infty)}$.

For $f \in \mathcal{A}_{[0,\infty)}$, by the definition of $T^{(q)}_{L_N}$ as an extension of T_{L_N}, $T^{(q)}_{L_N} f = T_{L_N} f = \sum_{\alpha \in L_N} \widehat{f}(\alpha) e_\alpha$ in $\mathcal{A}^{(q)}_{[0,\infty)}$.

Next $F_N \to F$ in $\mathcal{A}^q_{[0,\infty)}$ as $N \to \infty$ is proved. For $\epsilon > 0$ choose $f \in \mathcal{A}_{[0,\infty)}$ with $\|F - f\|_{q,[0,\infty)} \le \epsilon$ and $\widehat{f}(\alpha) = 0$ for all $\alpha \in \mathbb{R}$ with $\widehat{F}(\alpha) = 0$. Choose $N_0 \ge 1$ such that $\{\alpha \in \mathbb{R} \mid \widehat{f}(\alpha) \neq 0\} \subseteq L_{N_0}$. Then for $N \ge N_0$,

$$\|F - F_N\|_{q,[0,\infty)} \le \|F - f\|_{q,[0,\infty)} + \|T^{(q)}_{L_N} f - T^{(q)}_{L_N} F\|_{q,[0,\infty)} \le 2\epsilon.$$

It follows that $F_N^q \to F^q$ in $\mathcal{A}^1_{[0,\infty)}$ and thus $M_{[0,\infty)}(F_N^q) \to M_{[0,\infty)}(F^q)$ as $N \to \infty$. Now let $q \in \mathbb{N}$. Since (4.1) is absolutely convergent,

$$M_{[0,\infty)}(F_N^q) = \sum \frac{d(|n_1|)\cdots d(|n_q|)}{(2\pi\sqrt{2})^q |n_1 \cdots n_q|^{3/4}} e^{-i\pi(\mathrm{sign}(n_1) + \cdots \mathrm{sign}(n_q))/4},$$

where the sum runs through all $n_1, \ldots, n_q \in \mathbb{Z} \setminus \{0\}$ with $\mathrm{sign}\,(n_1)\sqrt{|n_1|} + \cdots + \mathrm{sign}\,(N_q)\sqrt{|n_q|} = 0$ and $K(n_1), \ldots, K(n_q) \le N$. Here $K(n)$ denotes the squarefree kernel of $|n|$.

The series where the condition on the squarefree kernels is removed is absolutely convergent. To see this split up the range of summation corresponding to the subscripts j for which the $K(|n_j|)$ have the same values. Using Besicovitch's result shows that the series has the majorant

$$\sum_{l=1}^{q} \sum_{T_1\dot\cup\cdots\dot\cup T_l=\{1,\ldots,q\}} \sum_{s_1,\ldots,s_l\in\mathbb{N}\,\text{squarefree, pairwise distinct}}$$
$$\sum_{r_1,\ldots,r_q\in\mathbb{Z}\setminus\{0\}:\,\sum_{\kappa\in T_\lambda} r_\kappa=0\,\text{for}\,1\le\lambda\le l} \frac{d(s_1r_1^2)\cdots d(s_qr_q^2)}{|s_1^{|T_1|}\cdots s_l^{|T_l|}r_1^2\cdots r_q^2|^{3/4}}.$$

For the innermost sum to be non-empty it is necessary that $|T_\lambda|\ge 2$ for all λ. Thus for fixed $T_1,\ldots,T_l$ the two innermost sums have the majorant

$$\sum_{s_1,\ldots,s_l\in\mathbb{N},\,r_1,\ldots,r_q\in\mathbb{Z}\setminus\{0\}} \frac{d(s_1r_1^2)\cdots d(s_qr_q^2)}{|s_1^2\cdots s_l^2r_1^2\cdots r_q^2|^{3/4}},$$

which is clearly convergent. Thus

$$M_{[0,\infty)}(F^q) = \sum_{\substack{n_1,\ldots,n_q\in\mathbb{Z}\setminus\{0\}:\\ \mathrm{sign}(n_1)\sqrt{|n_1|}+\cdots+\mathrm{sign}(n_q)\sqrt{|n_q|}=0}} \frac{d(|n_1|)\cdots d(|n_q|)}{(2\pi\sqrt{2})^q|n_1\cdots n_q|^{3/4}} e^{-\pi i(\mathrm{sign}(n_1)+\cdots+\mathrm{sign}(n_q))/4}.$$

Splitting up the range of summation corresponding to the subscripts j for which n_j has the same sign finishes the proof. □

In particular, for $q=3$,

$$M_{[0,\infty)}(F^3) = \frac{3}{16\pi^3}S(1,3)$$

and for $q=4$,

$$M_{[0,\infty)}(F^4) = \frac{3}{32\pi^4}S(2,4)$$

corresponding to the results in [11].

5. Transfer of M. Riesz' theorem

Let the assumptions of Theorem 1.6 be fulfilled. To prove (a), let $1<q<\infty$ and define

$$\Delta_K := \left|F - \sum_{|k|\le K}\widehat{F}(k)e_k\right|^q.$$

M. Riesz' theorem ([3], 12.10.1) states that

$$\int_0^1 \Delta_K(x)\,dx \to 0 \text{ as } K\to\infty. \tag{5.1}$$

Since the sequence $(\alpha n)_{n\in\mathbb{Z}}$ is uniformly distributed modulo 1 and Δ_K is Riemann integrable the Weyl criterion ([6], Chapter 1, Corollary 1.1) gives

$$\lim_{N\to\infty}\frac{1}{2N+1}\sum_{|n|\le N}\Delta_K(\{\alpha n\}) = \int_0^1 \Delta_K(x)\,dx, \quad K\in\mathbb{N}. \tag{5.2}$$

By definition, the left hand side equals $\big\|f-\sum_{|k|\le K}\widehat{F}(k)e_{k\alpha}\big\|_q^q$. Thus (5.1) and (5.2) give

$$\Big\|f-\sum_{|k|\le K}\widehat{F}(k)e_{k\alpha}\Big\|_q \to 0 \text{ as } K\to\infty. \tag{5.3}$$

This proves (a) and $\widehat{f}(\beta)=\widehat{F}(k)$ for $\beta\equiv k\alpha \bmod 1$ ($k\in\mathbb{Z}$) and $\widehat{f}(\beta)=0$ otherwise.

To prove (b), let $1<p<\infty$ and $g\in\mathcal{A}^p$. In (a), choose $1<q<\infty$ with $1/p+1/q=1$. Then for $K\in\mathbb{N}$,

$$\begin{aligned}\Big|M(f\overline{g})-\sum_{|k|\le K}\widehat{F}(k)\overline{\widehat{g}(k\alpha)}\Big| &= \Big|M\Big(\Big(f-\sum_{|k|\le K}\widehat{F}(k)e_{k\alpha}\Big)\overline{g}\Big)\Big|\\ &\le \Big\|f-\sum_{|k|\le K}\widehat{F}(k)e_{k\alpha}\Big\|_q \|g\|_p,\end{aligned}$$

which together with (5.3) proves (b). $\square$

Proof of Corollary 1.7. Take F to be the characteristic function of the interval $[a,b]$. Then

$$\widehat{F}(k)=\frac{e(-ka)-e(-kb)}{2\pi ik},\ k\in\mathbb{Z}\setminus\{0\},\quad \widehat{F}(0)=b-a.$$

Applying Theorem 1.6(b) gives the first result of Corollary 1.7. If $g\in\mathcal{D}^p$, then $\widehat{g}(k\alpha)=0$ for $k\in\mathbb{Z}\setminus\{0\}$ and thus the second result follows. $\square$

6. A Hausdorff-Young inequality

The classical Hausdorff-Young inequality is formulated for periodic trigonometric polynomials ([3], 13.5.1 or [4], Chap. 23). Fortunately the proof via the Riesz-Thorin Interpolation Theorem can be easily generalized to the present situation.

Proposition 6.1. *Let $1\le p\le 2$, $1/p+1/q=1$ ($q=u$ if $p=1$). Then for $f\in\mathcal{A}$,*

$$\|f\|_q\le\Big(\sum_{\alpha\in\mathbb{R}/\mathbb{Z}}|\widehat{f}(\alpha)|^p\Big)^{1/p}.$$

Corollary 6.2. *For $1\le p\le 2$, $1/p+1/q=1$, $f\in\mathcal{A}^q$,*

$$\|f\|_q\le\Big(\sum_{\alpha\in\mathbb{R}/\mathbb{Z}}|\widehat{f}(\alpha)|^p\Big)^{1/p}.$$

Proof of Theorem 1.8. If suffices to show that g is the limit of its Fourier series in $\mathcal{A}^q$. Let

$$A=\big\{\alpha\in\mathbb{R}/\mathbb{Z}\,\big|\,\widehat{g}(\alpha)\ne 0\big\}=\bigcup_{j=1}^{\infty}A_j \tag{6.1}$$

with $A_1\subseteq A_2\subseteq\cdots$ and $|A_j|<\infty$ for $j\ge 1$. Corollary 6.2 gives

$$\Big\|g-\sum_{\alpha\in A_j}\widehat{g}(\alpha)e_\alpha\Big\|_q\le\Big(\sum_{\alpha\in A\setminus A_j}|\widehat{g}(\alpha)|^p\Big)^{1/p}.$$

Since

$$\left(\sum_{\alpha\in\mathbb{R}/\mathbb{Z}} |\widehat{g}(\alpha)|^p\right)^{1/p} < \infty,$$

the right hand side converges to 0 as $j \to \infty$. □

7. Proof of Theorem 1.9

Define $\tilde{f}(n) := f(-n)$, $n \in \mathbb{Z}$. According to Theorem 2.1(b), $\tilde{f} \times g$ can be chosen to lie in $\mathcal{A}^u$ such that for $a \in \mathbb{Z} \setminus \{0\}$,

$$\tilde{f} \times g(-a) = \lim_{N\to\infty} C_N(\tilde{f}, g)(-a) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{|n|\le N} f(a+n)g(n). \tag{7.1}$$

On the other hand, by Theorem 2.1(d),

$$\widehat{\tilde{f} \times g}(\alpha) = \widehat{\tilde{f}}(\alpha)\widehat{g}(\alpha) \text{ for } \alpha \in \mathbb{R}/\mathbb{Z}. \tag{7.2}$$

From the definition of $\mathcal{B}^r$, Corollary 2.3 and Theorem 1.2 it follows that for $h \in \mathcal{A}^r$, $r \in [1,\infty) \cup \{0\}$,

$$h \in \mathcal{B}^r \Leftrightarrow \begin{array}{l} \widehat{h}(\alpha) = 0 \text{ for } \alpha \notin \mathbb{Q}/\mathbb{Z}, \\ \widehat{h}\left(\frac{a}{r}\right) = \widehat{h}\left(\frac{b}{r}\right) \text{ for } a,b \in \mathbb{Z},\ r \in \mathbb{N},\ (a,r) = (b,r) = 1. \end{array} \tag{7.3}$$

From (7.2), (7.3) and $f \in \mathcal{B}^p$, $g \in \mathcal{B}^q$, it follows that $\tilde{f} \times g \in \mathcal{B}^u$. Hildebrand's theorem ([10], Chap. 5, Theorem 1.2) gives

$$\tilde{f} \times g(-a) = \sum_{r\ge 1} a_r(\tilde{f} \times g)c_r(-a), \quad a \in \mathbb{Z} \setminus \{0\}. \tag{7.4}$$

(Hildebrand states the theorem for $\mathcal{B}^u$-functions on $\mathbb{N}$ rather than $\mathbb{Z}$ but with Proposition 4.2 it can be lifted to the present situation.) Finally, for $r \in \mathbb{N}$, (7.2) and (7.3) give

$$a_r(\tilde{f} \times g) = \frac{1}{\varphi(r)} M((\tilde{f} \times g)c_r) = \widehat{\tilde{f} \times g}\left(\frac{1}{r}\right) = \widehat{\tilde{f}}\left(\frac{1}{r}\right)\widehat{g}\left(\frac{1}{r}\right) = a_r(f)a_r(g). \tag{7.5}$$

From (7.1), (7.4) and (7.5) the theorem follows. □

References

[1] A.S. Besicovitch, Almost Periodic Functions, Dover Publications 1954

[2] A.S. Besicovitch, On the linear independence of fractional powers of integers, J. London Math. Soc. 15(1940), 3–6

[3] R.E. Edwards, Fourier Series, Volume I,II, Holt, Rinehart and Winston, 1967

[4] P.D.T.A. Elliott, Duality in Analytic Number Theory, Cambridge Tracts in Mathematics 122, Cambridge University Press 1997

[5] D.R. Heath-Brown, The distribution and moments of the error term in the Dirichlet divisor problem, Acta. Arith. 60(1992), 389–415

[6] L. Kuipers, H. Niederreiter, Uniform Distribution, Wiley-Interscience, 1974

[7] W. Maak, Fastperiodische Funktionen, Springer-Verlag, Berlin-Göttingen-Heidelberg 1950

[8] W. Schwarz, Evaluation of mean values of products of shifted arithmetical functions, Note Mat. 15(1997), 235–250

[9] W. Schwarz, Evaluation of mean values of products of shifted arithmetical functions II, Number Theory in Progress, Vol. 2 (Zakopane 1997), 1081–1098, de Gruyter, Berlin, 1999

[10] W. Schwarz, J. Spilker, Arithmetical Functions, London Mathematical Society LNS 184, Cambridge University Press 1994

[11] K.-M. Tsang, Higher-power moments of $\Delta(x)$, $E(t)$ and $P(x)$, Proc. London Math. Soc. 65(1992), 65–84

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