THE DISTRIBUTION OF CLASS NUMBERS OF PURE NUMBER FIELDS

MANFRED PETER

1. Introduction

Much is known about the statistical distribution of class numbers of binary quadratic forms and quadratic fields. Let $d \equiv 0, 1 \mod 4$ and d not a perfect square. Define $h(d)$ as the number of equivalence classes of primitive binary quadratic forms with discriminant d (and positive definite in case $d < 0$). For $d > 0$, let $\epsilon_d := (u_d + v_d\sqrt{d})/2$, where (u_d, v_d) is the fundamental solution of Pell's equation $u^2 - dv^2 = 4$. If d is a fundamental discriminant then $h(d)$ is also the class number of $\mathbb{Q}(\sqrt{d})$ in the narrow sense.

Gauß [8] conjectured and Mertens [13] and Siegel [20] later proved that

$$\sum_{0<d\leq x} h(d) \log \epsilon_d \sim \frac{\pi^2}{18\zeta(3)} x^{3/2}, \qquad \sum_{0>d\geq -x} h(d) \sim \frac{\pi}{18\zeta(3)} x^{3/2}.$$

Chowla and Erdös [4] proved that there is a continuous distribution function F such that for all $z \in \mathbb{R}$,

$$\lim_{x\to\infty} \frac{1}{x/2} \#\Big\{0 < d \leq x \,\Big|\, \frac{h(d)\log \epsilon_d}{d^{1/2}} \leq e^z\Big\} = F(z),$$

$$\lim_{x\to\infty} \frac{1}{x/2} \#\Big\{0 > d \geq -x \,\Big|\, \frac{h(d)\pi}{|d|^{1/2}} \leq e^z\Big\} = F(z).$$

Elliott [6] showed that $F \in C^\infty(\mathbb{R})$ and it has the characteristic function

$$\Psi(t) = \prod_p \Big(\frac{1}{p} + \frac{1}{2}\Big(1 - \frac{1}{p}\Big)\Big(1 - \frac{1}{p}\Big)^{-it} + \frac{1}{2}\Big(1 - \frac{1}{p}\Big)\Big(1 + \frac{1}{p}\Big)^{-it}\Big), \quad t \in \mathbb{R}.$$

Barban [1] proved that for $q \in \mathbb{N}$, the $q - th$ moment β_q of $F(\log z)$ exists and that

$$\lim_{x\to\infty} \frac{1}{x/2} \sum_{0<d\leq x} \Big(\frac{h(d)\log \epsilon_d}{d^{1/2}}\Big)^q = \beta_q = \sum_{n\geq 1} \frac{\varphi(n) d_q(n^2)}{2n^3},$$

$$\lim_{x\to\infty} \frac{1}{x/2} \sum_{0>d\geq -x} \Big(\frac{h(d)\pi}{|d|^{1/2}}\Big)^q = \beta_q,$$

where φ is Euler's totient function and $d_q(n)$ is the number of ways one can write n as a product of q positive integers. For all these results, error term estimates can be given (see [3], [9], [19], [21], [24]).

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It seems that for number fields of higher degree, no analoguous results are known. The Brauer-Siegel Theorem (see, e.g., [12], Chapter XVI) gives a rough idea of the size of the class number times the regulator: Let k range over a sequence of number fields which are galois over $\mathbb{Q}$ such that $n/\log d \to 0$, where $n := [k : \mathbb{Q}]$ is the degree and $d = d_{k/\mathbb{Q}}$ is the absolute discriminant of k. Let h_k be the class number of k and R_k its regulator. Then

$$\frac{\log(h_k R_k)}{\log d^{1/2}} \to 1.$$

When looking for more precise information on the value distribution of

$$\frac{h_k R_k}{d^{1/2}},$$

we run into the problem of how to effectively parametrize number fields. This problem is avoided in the present paper by choosing a special class of number fields: Let l be a fixed rational prime and

$$S_l := \{m \in \mathbb{N} \setminus \{1\} \,|\, m \text{ is } l\text{–power–free}\}.$$

For $m \in S_l$, define the pure number field $k_m := \mathbb{Q}(\sqrt[l]{m})$ where the radical is choosen in $\mathbb{R}^+$. Let $r(m) := \operatorname{res}_{s=1} \zeta_{k_m}(s)$ where ζ_{k_m} is the Dedekind zeta function of k_m. Then

$$r(m) = \frac{h_{k_m} R_{k_m}}{d_{k_m}^{1/2}}\, c(l), \quad c(l) = \left\{ \begin{array}{ll} 2, & l = 2, \\ (2\pi)^{(l-1)/2}, & l \geq 3, \end{array} \right\}$$

and $d_{k_m} \asymp K(m)^{l-1}$, where $K(m)$ is the squarefree kernel of m. For $m \in \mathbb{N} \setminus S_l$, define $r(m) := 0$.

Theorem. *There is a distribution function $F \in C^\infty(\mathbb{R})$ such that for all $z \in \mathbb{R}$,*

$$\lim_{x\to\infty} \frac{\#\{m \in S_l \,|\, m \leq x,\, r(m) \leq e^z\}}{\#\{m \in S_l \,|\, m \leq x\}} = F(z).$$

Furthermore,

$$\lim_{x\to\infty} \frac{1}{\#\{m \in S_l \,|\, m \leq x\}} \sum_{m \in S_l \colon m \leq x} r(m)^q = \int_{\mathbb{R}^+} z^q \, dF(\log z)$$

for all $q \in \mathbb{N}$. The characteristic function $\Psi(t)$ of F is an Euler product whose factors depend on $t \in \mathbb{R}$.

The idea of proof is as follows: For $q \geq 1$, the function r is approximated in the q–th mean by functions R_P, $P \in \mathbb{N}$, such that

$$\|r - R_P\|_q \to 0 \text{ as } P \to \infty.$$

Here

$$\|f\|_q := \Big(\limsup_{x\to\infty} \frac{1}{x} \sum_{m \leq x} |f(m)|^q \Big)^{1/q} \in [0, \infty]$$

for $f : \mathbb{N} \to \mathbb{C}$. This step relies heavily on a zero density estimate of Kawada [10]. The R_P are partial products of Euler products derived from ζ_{k_m}. They are almost periodic which follows from the relation between the splitting of rational primes p in k_m and the splitting of $X^l - m$ in $\mathbb{F}[X]$ and $\mathbb{Q}_p^{\text{unram}}[X]$. Here $\mathbb{Q}_p^{\text{unram}}$ is the maximal unramified extension of $\mathbb{Q}_p$.

Since almost periodic functions have limit distributions, a standard argument shows the same for r. In fact the procedure in this last step is somewhat different since we also want to show the smoothness of F.

$c_1, c_2, \ldots$ will denote positive constants depending on the parameters given in parentheses. By ϵ we denote an arbitrary positive real.

2. Splitting of rational primes in k_m

The material in this section belongs to concrete algebraic number theory and is not new. For the convenience of the reader the relevant results are given in modern language and proofs are scetched.

For two polynomials f, g denote the discriminant of f by $\operatorname{discr}(f)$ and the resultant of f and g by $\mathrm{R}(f, g)$.

Proposition 2.1. *Let $\mathfrak{o}$ be a complete discrete valuation ring with characteristic 0 and maximal ideal $\mathfrak{p} = \pi\mathfrak{o}$. Assume that the monic separable polynomial $f \in \mathfrak{o}[X]$ has the prime decomposition $f = f_1 \cdots f_r$ in $\mathfrak{o}[X]$. Let $a \in \mathbb{N}_0$ with $\pi^a \| \operatorname{discr}(f)$ and for $1 \le i < j \le r$, let $\rho_{ij} \in \mathbb{N}_0$ with $\pi^{\rho_{ij}} \| \mathrm{R}(f_i, f_j)$. Then $\rho' := \sum_{i<j} \rho_{ij} \le a/2$. For all monic $g \in \mathfrak{o}[X]$ with $\deg g = \deg f$ and $g \equiv f \mod \pi^{a+1}$, there is a prime decomposition $g = g_1 \cdots g_r$ with*

$$\deg g_i = \deg f_i,\ g_i \equiv f_i \mod \pi^{a+1-\rho'},\ 1 \le i \le r.$$

Proof. This proposition rests on a generalization of Hensel's lemma and its essence is contained in [16]. For the formulation in terms of valuation rings, see [2]. □

For a rational prime p let $\mathbb{Q}_p$ be the field of p–adic numbers and $\overline{\mathbb{Q}}_p$ an algebraic closure of $\mathbb{Q}_p$.

Proposition 2.2. *Let $\alpha \in \mathbb{C}$ be a zero of the monic irreducible polynomial $f \in \mathbb{Q}[X]$ and define $K := \mathbb{Q}(\alpha)$. Let $p\,\mathcal{O}_K = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r}$ be the prime ideal decomposition of the rational prime p in K. For $1 \le i \le r$, let d_i be the residue class degree of $\mathfrak{p}_i$. Set $n_i := e_i d_i$ and let $\xi_i \in \overline{\mathbb{Q}}_p$ be a $(p^{n_i} - 1)$–st primitive root of unity. There is a prime decomposition $f = f_1 \cdots f_r$ in $\mathbb{Q}_p[X]$, where the f_i are monic of degree n_i and different from each other. Each f_i has a prime decomposition $f_i = g_{i1} \cdots g_{id_i}$ in $\mathbb{Q}_p(\xi_i)[X]$, where each g_{ij} is monic of degree e_i.*

Proof. Let $f = f_1 \cdots f_{r'}$ be a prime decomposition in $\mathbb{Q}_p[X]$ with monic factors. Since f is separable, all the f_i are different. Let $\alpha_i \in \overline{\mathbb{Q}}_p$ be a zero of f_i for $1 \le i \le r'$. Let v_p be the p–adic valuation on $\overline{\mathbb{Q}}_p$. It is well known (see, e.g., Neukirch [15], Chapter II, Theorem 8.2) that $r' = r$ and the enumeration can be choosen such that for $1 \le i \le r$ there is a homomorphism $\tau_i : K \to \overline{\mathbb{Q}}_p$ with $\tau_i(\alpha) = \alpha_i$ and $v_p \circ \tau_i$ is the continuation of v_p to K that belongs to $\mathfrak{p}_i$. Thus $\mathbb{Q}_p(\alpha_i)$ is isomorphic to a completion of K with respect to $v_p \circ \tau_i$, and

$$\deg f_i = [\mathbb{Q}_p(\alpha_i) : \mathbb{Q}_p] = e_i d_i = n_i.$$

If $d \in \mathbb{N}$ and $\xi \in \overline{\mathbb{Q}}_p$ is a $(p^d - 1)$–st primitive root of unity then $\mathbb{Q}_p(\xi)$ is the uniquely determined unramified extension of $\mathbb{Q}_p$ in $\overline{\mathbb{Q}}_p$ of degree d (see, e.g., [11], Chapter III, Section 3). Let $\xi_i' \in \overline{\mathbb{Q}}_p$ be a primitive $(p^{d_i} - 1)$–st root of unity. Then $\mathbb{Q}_p(\xi_i') \subseteq \mathbb{Q}_p(\alpha_i)$. Thus $\mathbb{Q}_p(\alpha_i)/\mathbb{Q}_p(\xi_i')$ is completely ramified of degree e_i. Since $d_i | n_i$, we have $p^{d_i} - 1 | p^{n_i} - 1$

and thus $\mathbb{Q}_p(\xi_i') \subseteq \mathbb{Q}_p(\alpha_i) \cap \mathbb{Q}_p(\xi_i) =: L$. Since $L/\mathbb{Q}_p(\xi_i')$ is unramified and completely ramified, we have $\mathbb{Q}_p(\xi_i') = L = \mathbb{Q}_p(\alpha_i) \cap \mathbb{Q}_p(\xi_i)$. Since $\mathbb{Q}_p(\xi_i)/\mathbb{Q}_p$ is finite and galois the Translation Theorem gives

$$[\mathbb{Q}_p(\xi_i, \alpha_i) : \mathbb{Q}_p(\xi_i)] = [\mathbb{Q}_p(\alpha_i) : \mathbb{Q}_p(\xi_i')] = e_i.$$

Let $f_i = g_{i1} \cdots g_{id}$ be the decomosition of f_i in $\mathbb{Q}_p(\xi_i)$ into monic irreducible polynomials. We can assume $g_{i1}(\alpha_i) = 0$. Then g_{i1} is the minimal polynomial of α_i over $\mathbb{Q}_p(\xi_i)$ and thus $\deg g_{i1} = e_i$. Let $1 \leq j \leq d$ and $\beta \in \overline{\mathbb{Q}}_p$ a zero of g_{ij}. Then $f_i(\beta) = 0$ and thus α_i and β are conjugate of $\mathbb{Q}_p$. Let σ be a $\mathbb{Q}_p$–automorphism of $\overline{\mathbb{Q}}_p$ with $\sigma(\alpha_i) = \beta$. If g_{i1}^σ is the image of g_{i1} under σ, we have $g_{i1}^\sigma(\beta) = \sigma(g_{i1}(\alpha_i)) = 0$. Since $\mathbb{Q}_p(\xi_i)/\mathbb{Q}_p$ is normal, we have $g_{i1}^\sigma \in \mathbb{Q}_p(\xi_i)[X]$ and thus g_{i1}^σ is the minimal polynomial of β over $\mathbb{Q}_p(\xi_i)$. Therefore $g_{ij} = g_{i1}^\sigma$ and in particular $\deg g_{ij} = \deg g_{i1} = e_i$. Since this holds for all $1 \leq j \leq d$, we finally have $d \cdot e_i = n_i$, e.g. $d = d_i$. □

Proposition 2.3. *Let $f \in \mathbb{Z}[X]$ be monic and irreducible and $\alpha \in \mathbb{C}$ a zero of f. Define $K := \mathbb{Q}(\alpha)$ and let $p\,\mathcal{O}_K = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r}$ be the prime ideal decomposition of the rational prime p in K. For $1 \leq i \leq r$ let d_i be the residue class degree of $\mathfrak{p}_i$. Let $a \in \mathbb{N}_0$ with $p^a \| \operatorname{discr}(f)$. Then for every monic irreducible $g \in \mathbb{Z}[X]$ with $\deg g = \deg f$ and $g \equiv f \mod p^{a+1}$ the following holds: If $\beta \in \mathbb{C}$ is a zero of g and $K' := \mathbb{Q}(\beta)$, then the prime ideal decomposition of p in K' is of the form*

$$p\,\mathcal{O}_{K'} = \mathfrak{q}_1^{e_1} \cdots \mathfrak{q}_r^{e_r},$$

and for $1 \leq i \leq r$ the residue class degree of $\mathfrak{q}_i$ is d_i.

Proof. We combine Propositions 2.1 and 2.2 several times. Let $p\,\mathcal{O}_{K'} = \mathfrak{q}_1^{e_1'} \cdots \mathfrak{q}_s^{e_s'}$ be the prime ideal decomposition of p in K' and d_j' the residue class degree of $\mathfrak{q}_j$. Let $\xi_i \in \overline{\mathbb{Q}}_p$ be a primitive $(p^{n_i} - 1)$–st root of unity where $n_i := e_i d_i$. By Proposition 2.2 we have a prime decomposition $f = f_1 \cdots f_r$ in $\mathbb{Q}_p[X]$ where f_i is monic of degree n_i. Gauß' Lemma gives $f_1, \ldots, f_r \in \mathbb{Z}_p[X]$. Let $\rho_{ij} \in \mathbb{N}_0$ with $p^{\rho_{ij}} \| \mathrm{R}(f_i, f_j)$ and $\rho' := \sum_{i<j} \rho_{ij}$. From Proposition 2.1 it follows that there is a prime decomposition $g = g_1 \cdots g_r$ in $\mathbb{Z}_p[X]$ with $\deg g_i = \deg f_i$ and $g_i \equiv f_i \mod p^{a'}$ for $1 \leq i \leq r$, where $a' := a + 1 - \rho'$. By Proposition 2.2 again it follows that $s = r$ and, after some reordering, $n_i' := e_i' d_i' = n_i$ for $1 \leq i \leq r$. A well known theorem gives

$$\operatorname{discr}(f) = \prod_{i=1}^{r} \operatorname{discr}(f_i) \prod_{i \neq j} \mathrm{R}(f_i, f_j)$$

and thus $a = \sum_{i=1}^{r} \operatorname{ord}_p \operatorname{discr}(f_i) + 2\rho'$. Therefore $a' > \operatorname{ord}_p \operatorname{discr}(f_i)$ for $1 \leq i \leq r$. Fix $1 \leq i \leq r$ and let $\mathfrak{o}_i$ be the valuation ring of $K_i := \mathbb{Q}_p(\xi_i)$. Since $K_i/\mathbb{Q}_p$ is unramified, the maximal ideal of $\mathfrak{o}_i$ is $p\mathfrak{o}_i$. By Proposition 2.2 there is a prime decomposition $f_i = f_{i1} \cdots f_{id_i}$ in $K_i[X]$ where each f_{ij} is monic of degree e_i. Gauß' Lemma shows that $f_{ij} \in \mathfrak{o}_i[X]$ for $1 \leq j \leq d_i$. Since $g_i \equiv f_i \mod p^{\operatorname{ord}_p \operatorname{discr}(f_i)+1}$ and $\deg g_i = \deg f_i$, Proposition 2.1 gives a prime decomposition $g_i = g_{i1} \cdots g_{id_i}$ in $\mathfrak{o}_i[X]$ with $\deg g_{ij} = \deg f_{ij} = e_i$ for $1 \leq i \leq d_i$. A final application of Proposition 2.2 gives $e_i' = e_i$ and $d_i' = n_i'/e_i' = n_i/e_i = d_i$. □

Proposition 2.4. *Let l be a rational prime and $m \in S_l$. Then $[k_m : \mathbb{Q}] = l$ and for all rational primes $p \neq l$, we have*

$$\begin{aligned}\#\{\mathfrak{p}|p \,|\, \mathfrak{p} \text{ is a prime ideal in } k_m \text{ with residue class degree } 1\} \\ = \#\{x \mod p \,|\, x^l \equiv m \mod p\}. \qquad (2.1)\end{aligned}$$

Proof. Let p be a rational prime with $p|m$. Let $a \in \mathbb{N}$ with $p^a \| m$. Since m is l–power–free, we have $l \not| a$ and there are $x, y \in \mathbb{Z}$ with $ax + ly = 1$, $x > 0$. Define $m' := (mp^{-a})^x p \in \mathbb{N}$. Then $m' = m^x p^{yl}$, $m = (m')^a (mp^{-a})^{yl}$ and thus $k_m = k_{m'}$.

Since $m \neq 1$ there is a rational prime p as above. Since $p \| m'$, the polynomial $X^l - m' \in \mathbb{Z}[X]$ is Eisensteinian with respect to p and thus irreducible. Therefore $[k_m : \mathbb{Q}] = [k_{m'} : \mathbb{Q}] = l$.

Now let $p \neq l$ be an arbitrary rational prime. If $p|m$, Proposition 4.18 of [14] is applicable and gives $p\,\mathcal{O}_{k_m} = p\,\mathcal{O}_{k_{m'}} = \mathfrak{p}^l$ with some prime ideal $\mathfrak{p}$ of k_m with residue class degree 1. Thus the left hand side of (2.1) equals 1. The same holds for the right hand side since $p|m$. Now assume $p \not| m$. The absolute discriminant of the elements $\sqrt[l]{m}^j$, $0 \leq j \leq l-1$, is $d = l^l m^{l-1}$ and therefore $p \not| d$. For the index $t := [\mathcal{O}_{k_m} : \mathbb{Z}[\sqrt[l]{m}]]$, we have $d = t^2 d_{k_m/\mathbb{Q}}$ where $d_{k_m/\mathbb{Q}}$ is the absolute discriminant of k_m. Thus $p \not| t$ and Theorem 4.12 of [14] is applicable. Let $X^l - m \equiv f_1 \cdots f_r \mod p$ with monic irreducible polynomials $f_1, \ldots, f_r \mod p$. Since $p \not| d$, the f_i are different from each other. Therefore $p\,\mathcal{O}_{k_m} = \mathfrak{p}_1 \cdots \mathfrak{p}_r$ with different prime ideals $\mathfrak{p}_i$ of k_m; each $\mathfrak{p}_i$ has residue class degree $\deg f_i$. Thus the left hand side of (2.1) equals $\#\{1 \leq i \leq r \,|\, \deg f_i = 1\}$ which is the number of zeros of $X^l - m \mod p$. □

3. Some auxiliary functions

Let p be a rational prime. For $m \in S_l$ define

$$\begin{aligned}&\chi(m,p) := \#\{\mathfrak{p}|p \,|\, \mathfrak{p} \text{ is a prime ideal of } k_m \text{ with } f(\mathfrak{p}/p) = 1\} - 1, \\ &\gamma_{p1}(m) := \left(1 - \frac{1}{p}\right)^{-\chi(m,p)}, \quad \gamma_{p2}(m) := \prod_{\mathfrak{p}|p:\, f(\mathfrak{p}/p) \geq 2} \left(1 - \frac{1}{p^{f(\mathfrak{p}/p)}}\right)^{-1}, \\ &\gamma_p(m) := \gamma_{p1}(m)\gamma_{p2}(m),\end{aligned}$$

where $f(\mathfrak{p}/p) := [\mathcal{O}_{k_m}/\mathfrak{p} : \mathbb{Z}/p\mathbb{Z}]$ is the residue class degree of $\mathfrak{p}$. For $m \in \mathbb{N} \setminus S_l$, define $\gamma_{p1}(m) := \gamma_{p2}(m) := \gamma_p(m) := 0$. For $b \in \mathbb{N}_0$, let $\mathcal{R}_{pb}$ be a complete system of representatives of the coprime residue classes modulo $p^{b(l-2)+l+1}$. From Dirichlet's prime number theorem it follows that we can assume $\mathcal{R}_{pb} \subseteq S_l$.

Lemma 3.1. *For p a rational prime, $0 \leq b \leq l-1$, $m \in \mathcal{R}_{pb}$ and $m' \in S_l$ with $m' \equiv p^b m \mod p^{b(l-1)+l+1}$, we have $\gamma_{pi}(m') = \gamma_{pi}(p^b m)$ for $i = 1, 2$.*

Proof. For $m'' := p^b m \in S_l$, the absolute discriminant of $X^l - m''$ is $d = l^l (m'')^{l-1}$ and hence $p^{b(l-1)+\delta_p} \| d$, where $\delta_p = l$ for $p = l$ and $\delta_p = 0$ otherwise. Since $m' \equiv m'' \mod p^{b(l-1)+\delta_p+1}$, Proposition 2.3 shows that the invariants connected with the decomposition of p in $k_{m'}$ and $k_{m''}$ are the same after some reordering. Therefore $\gamma_{pi}(m') = \gamma_{pi}(m'')$. □

Lemma 3.2. *The density of S_l is*

$$d(S_l) := \lim_{x\to\infty} \frac{1}{x}\ \#\{m \in S_l \,|\, m \le x\} = \zeta(l)^{-1}.$$

For $a, q \in \mathbb{N}$ coprime, we have

$$\lim_{x\to\infty} \frac{1}{x}\ \#\{m \in S_l \,|\, m \le x,\, m \equiv a \mod q\} = \frac{1}{q}\prod_{p|q}\Big(1-\frac{1}{p^l}\Big)^{-1} d(S_l).$$

Proof. Let χ be a Dirichlet character modulo q. For $x > 0$, define

$$S_\chi(x) := \sum_{m\in S_l:\, m\le x} \chi(m).$$

Then

$$T_\chi(x) := \sum_{d\le \sqrt[l]{x}} \chi(d^l) S_\chi\Big(\frac{x}{d^l}\Big) = \sum_{n\le x}\chi(n) = \left\{ \begin{array}{ll} \varphi(q)x/q + O(q), & \chi = \chi_0, \\ O(q), & \chi \ne \chi_0. \end{array} \right\}$$

Möbius inversion gives

$$S_\chi(x) = \sum_{d\le \sqrt[l]{x}} \mu(d)\chi(d^l) T_\chi\Big(\frac{x}{d^l}\Big) = \left\{ \begin{array}{ll} \dfrac{\varphi(q)}{q}\displaystyle\prod_{p\nmid q}\Big(1-\frac{1}{p^l}\Big)x + O(q\sqrt[l]{x}), & \chi = \chi_0, \\ O(q\sqrt[l]{x}), & \chi \ne \chi_0. \end{array} \right\}$$

Now the orthogonality relation for characters gives

$$\#\{m \in S_l \,|\, m \le x,\, m \equiv a \mod q\} = \frac{1}{\varphi(q)}\sum_{\chi \mod q} S_\chi(x)\overline{\chi}(a) = \frac{1}{q}\prod_{p\nmid q}\Big(1-\frac{1}{p^l}\Big)x + O(q\sqrt[l]{x}).$$

□

For $P \ge 2$, define the function

$$R_P(m) := \prod_{p\le P}\gamma_p(m), \quad m \in \mathbb{N}.$$

For $x \ge 1$, define $S_l(x) := \#\{m \in S_l \,|\, m \le x\}$. Furthermore, define the distribution function

$$F_{P,x}(z) := \frac{1}{S_l(x)}\ \#\{m \in S_l \,|\, m \le x,\, R_P(m) \le e^z\}, \quad z \in \mathbb{R}.$$

Let $\Psi_{P,x}$ be the characteristic function of $F_{P,x}$. For p a rational prime and $t \in \mathbb{R}$, define

$$\psi(p,t) := \Big(1-\frac{1}{p^l}\Big)^{-1}\sum_{0\le b\le l-1}\frac{1}{p^{b(l-1)+l+1}}\sum_{m\in\mathcal{R}_{p^b}}\gamma_p(p^b m)^{it}.$$

Finally, set

$$\Psi_P(t) := \prod_{p\le P}\psi(p,t).$$

Lemma 3.3. *For all $P \ge 2$, we have $\lim_{x\to\infty}\Psi_{P,x}(t) = \Psi_P(t)$ uniformly in $t \in \mathbb{R}$.*

Proof. For $t \in \mathbb{R}$, we have

$$\begin{aligned}\Psi_{P,x}(t) &= \int_{\mathbb{R}} e^{itz} dF_{P,x}(z) = \frac{1}{S_l(x)} \sum_{m \in S_l \colon m \le x} R_P(m)^{it} \\ &= \frac{1}{S_l(x)} \sum_{0 \le b_p \le l-1 \, (p \le P)} \quad \sum_{m \in S_l \colon m \le x, \, \mathrm{ord}_p \, m = b_p \, (p \le P)} R_P(m)^{it}. \end{aligned} \tag{3.1}$$

The inner sum equals

$$\sum_{m_p \in \mathcal{R}_{pb_p} \, (p \le P)} \quad \sum_{m \in S_l \colon m \le x, \, m \equiv p^{b_p} m_p \mod p^{b_p(l-1)+l+1} \, (p \le P)} R_P(m)^{it}. \tag{3.2}$$

Fix m_p for $p \le P$. It follows from Lemmas 3.1 and 3.2 that the inner sum in (3.2) equals

$$\begin{aligned}&\prod_{p \le P} \gamma_p(p^{b_p} m_p)^{it} \; \#\Big\{ m' \in S_l \, \Big| \, m' \le x \prod_{p \le P} p^{-b_p}, \, m' \equiv a \mod q \Big\} \\ &\qquad \sim \prod_{p \le P} \gamma_p(p^{b_p} m_p)^{it} \; \frac{1}{q} \prod_{p \mid q} \Big(1 - \frac{1}{p^l}\Big)^{-1} d(S_l) \frac{x}{\prod_{p \le P} p^{b_p}}, \end{aligned} \tag{3.3}$$

where $q := \prod_{p \le P} p^{b_p(l-2)+l+1}$ and $a \in \mathbb{N}$ with $\gcd(a, q) = 1$ depends on the b_p and m_p. Putting (3.1), (3.2) and (3.3) together we see that for fixed $P \ge 2$, we have

$$\begin{aligned}\Psi_{P,x}(t) =& \frac{1}{S_l(x)} \sum_{0 \le b_p \le l-1 \, (p \le P)} \quad \sum_{m_p \in \mathcal{R}_{pb_p} \, (p \le P)} \\ & \prod_{p \le P} \gamma_p(p^{b_p} m_p)^{it} \prod_{p \le P} p^{-b_p(l-1)-l-1} \Big(1 - \frac{1}{p^l}\Big)^{-1} d(S_l) x + o(1) \\ =& \Psi_P(t) + o(1)\end{aligned}$$

as $x \to \infty$ uniformly in $t \in \mathbb{R}$. □

Lemma 3.4. *Let $c > 0$. For all $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in [-c, c]$ with $\sum_{j=1}^n a_j = 0$, we have*

$$\frac{1}{n} \Big| \sum_{j=1}^n e^{ia_j} \Big| \le \exp\Big(- \frac{1}{2n} \sum_{j=1}^n a_j^2 + O_c\Big(\frac{1}{n} \sum_{j=1}^n |a_j|^3 \Big) \Big).$$

Proof. The power series expansion of the exponential function gives

$$\begin{aligned}\frac{1}{n} \Big| \sum_{j=1}^n e^{ia_j} \Big| &= \frac{1}{n} \Big| \sum_{j=1}^n \big(1 + ia_j + \frac{(ia_j)^2}{2} + O_c(|a_j|^3) \big) \Big| \\ &= \Big| 1 - \frac{1}{2n} \sum_{j=1}^n a_j^2 + O_c\Big(\frac{1}{n} \sum_{j=1}^n |a_j|^3 \Big) \Big|.\end{aligned}$$

Together with Hölder's inequality it follows that

$$\frac{1}{n} \Big| \sum_{j=1}^n e^{ia_j} \Big| \exp\Big(\frac{1}{2n} \sum_{j=1}^n a_j^2 \Big)$$

$$= \Big|1 - \frac{1}{2n}\sum_{j=1}^n a_j^2 + O_c\Big(\frac{1}{n}\sum_{j=1}^n |a_j|^3\Big)\Big| \cdot \Big|1 + \frac{1}{2n}\sum_{j=1}^n a_j^2 + O_c\Big(\Big(\frac{1}{n}\sum_{j=1}^n a_j^2\Big)^2\Big)\Big|$$
$$\le 1 + O_c\Big(\Big(\frac{1}{n}\sum_{j=1}^n a_j^2\Big)^2 + \frac{1}{n}\sum_{j=1}^n |a_j|^3\Big) \le 1 + O_c\Big(\frac{1}{n}\sum_{j=1}^n |a_j|^3\Big)$$
$$\le \exp\Big(O_c\Big(\frac{1}{n}\sum_{j=1}^n |a_j|^3\Big)\Big).$$

□

Lemma 3.5. (1) *There is some $c \ge 1$ such that for all $|t| \ge c$, $p \ge c(|t|+1)$, $p \equiv 1 \mod l$, we have*
$$|\psi(p,t)| \le \exp\Big(-\frac{l-1}{4p^2}t^2\Big).$$
(2) *For all $c' \ge 1$, $|t| \le c'$, $p \ne l$ a rational prime, we have*
$$\psi(p,t) = 1 + O_{c'}\Big(\frac{1}{p^2}\Big).$$

Proof. Let $p \ne l$ be a rational prime, $c'' \ge 1$ and $|t| \le c''p$. Then
$$\psi(p,t)\Big(1-\frac{1}{p^l}\Big) = \frac{1}{p^{l+1}}\sum_{m\in\mathcal{R}_{p0}} \gamma_p(m)^{it} + \frac{1}{p^{2l}}\sum_{m\in\mathcal{R}_{p1}} \gamma_p(pm)^{it}$$
$$+ O\Big(\sum_{2\le b\le l-1} \frac{1}{p^{b(l-1)+l+1}} p^{b(l-2)+l+1}\Big).$$
The error term is $O(p^{-2})$. For $m \in S_l$, we have
$$\gamma_{p2}(m)^{it} = \exp\Big(-it\sum_{\mathfrak{p}|p\colon f(\mathfrak{p}/p)\ge 2} \log\Big(1-\frac{1}{p^{f(\mathfrak{p}/p)}}\Big)\Big)$$
$$= \exp\Big(\sum_{\mathfrak{p}|p\colon f(\mathfrak{p}/p)\ge 2} O\Big(\frac{|t|}{p^2}\Big)\Big) = \exp\Big(O\Big(\frac{|t|}{p^2}\Big)\Big)$$
$$= 1 + O_{c''}\Big(\frac{|t|}{p^2}\Big).$$
Thus
$$\psi(p,t)\Big(1-\frac{1}{p^l}\Big) = \frac{1}{p^{l+1}}\sum_{m\in\mathcal{R}_{p0}}\Big(1-\frac{1}{p}\Big)^{-\chi(m,p)it} + \frac{1}{p^{2l}}\sum_{m\in\mathcal{R}_{p1}}\Big(1-\frac{1}{p}\Big)^{-\chi(pm,p)it} + O_{c''}\Big(\frac{|t|+1}{p^2}\Big).$$
Since $p \ne l$, Proposition 2.4 gives $\chi(m,p) = \rho(m,p)$ for $m \in S_l$, where
$$\rho(m,p) := \#\{x \mod p \,|\, x^l \equiv m \mod p\} - 1, \quad m \in \mathbb{Z}.$$
The function $\rho(\cdot,p)$ is p–periodic and
$$\sum_{m \mod p} \rho(m,p) = 0. \tag{3.4}$$

Thus

$$\begin{aligned}\psi(p,t) =&\frac{1}{p^{l+1}}\sum_{m \bmod p\colon p\nmid m}\Big(1-\frac{1}{p}\Big)^{-\rho(m,p)it}p^l\\&+\frac{1}{p^{2l}}\Big(1-\frac{1}{p}\Big)^{-\rho(0,p)it}p^{2l-2}(p-1)+O_{c''}\Big(\frac{|t|+1}{p^2}\Big)+\frac{1}{p^l}\psi(p,t)\\=&\frac{1}{p}\sum_{m \bmod p}\Big(1-\frac{1}{p}\Big)^{-\rho(m,p)it}+O_{c''}\Big(\frac{|t|+1}{p^2}\Big).\end{aligned}\tag{3.5}$$

(1) Let $c\ge 1$ and assume $|t|\ge c$, $p\ge c(|t|+1)$, $p\equiv 1 \bmod l$. The following O–constants do not depend on c. By Lemma 3.4 it follows from (3.5) and (3.4) that

$$\begin{aligned}|\psi(p,t)| \le&\Big|\frac{1}{p}\sum_{m \bmod p}e^{-it\rho(m,p)\log(1-1/p)}\Big|+O\Big(\frac{|t|}{p^2}\Big)\\\le&\exp\Big(-\frac{1}{2p}\sum_{m \bmod p}t^2\rho(m,p)^2\log^2\Big(1-\frac{1}{p}\Big)\\&+O\Big(\frac{1}{p}\sum_{m \bmod p}|t|^3\rho(m,p)^3\Big|\log\Big(1-\frac{1}{p}\Big)\Big|^3\Big)\Big)+O\Big(\frac{|t|}{p^2}\Big)\end{aligned}$$

since

$$\Big|-it\rho(m,p)\log\Big(1-\frac{1}{p}\Big)\Big|\le |t|(l+1)\frac{1}{p-1}\le l+1.$$

From $l|p-1$ it follows that $\mathbb{F}_p^*$ has a cyclic subgroup of order l. Thus the kernel of the homomorphism $\mathbb{F}_p^*\to\mathbb{F}_p^*$, $x\mapsto x^l$, has order l. Therefore $\rho(0,p)=0$, $\rho(m,p)=l-1$ for $(p-1)/l$ elements of $\mathbb{F}_p^*$ and $\rho(m,p)=-1$ for the rest of them. This gives

$$\sum_{m \bmod p}\rho(m,p)^2=(p-1)(l-1)$$

and therefore

$$|\psi(p,t)|\le\exp\Big(-\frac{l-1}{2p^2}t^2+O\Big(\frac{|t|^3}{p^3}\Big)\Big)+O\Big(\frac{|t|}{p^2}\Big).$$

For $x,y\in\mathbb{R}$, $|x|,|y|\le c_4$, we have $\exp(x)+y\le\exp(x+y+O_{c_4}(x^2))$. This gives

$$\begin{aligned}|\psi(p,t)|\le&\exp\Big(-\frac{l-1}{2p^2}t^2+O\Big(\frac{|t|^3}{p^3}\Big)+O\Big(\frac{|t|}{p^2}\Big)+O\Big(\frac{|t|^4}{p^4}\Big)\Big)\\=&\exp\Big(-\frac{l-1}{2p^2}t^2+O\Big(\frac{t^2}{p^2c}\Big)\Big).\end{aligned}$$

Choosing $c\ge 1$ large enough gives $|\psi(p,t)|\le\exp(-(l-1)t^2/(4p^2))$.

(2) Now let $c'\ge 1$ be arbitrary, $|t|\le c'$ and $p\ne l$. Since

$$\Big|-it\rho(m,p)\log\Big(1-\frac{1}{p}\Big)\Big|\le|t|(l+1)\frac{1}{p-1}\le c'(l+1),$$

Taylor expansion in (3.5) together with (3.4) gives

$$\psi(p,t) = \frac{1}{p} \sum_{m \mod p} \Big(1 - it\rho(m,p)\log\Big(1-\frac{1}{p}\Big) + O_{c'}\Big(\frac{|t|^2}{p^2}\Big)\Big) + O_{c'}\Big(\frac{1}{p^2}\Big)$$
$$= 1 + O_{c'}\Big(\frac{1}{p^2}\Big).$$

□

Lemma 3.6. (1) *The infinite product* $\Psi(t) := \prod_p \psi(p,t)$ *converges uniformly for* t *in any bounded subset of* $\mathbb{R}$*. As* $|t| \to \infty$*, we have*

$$\Psi(t) \ll \exp\Big(-\frac{c_5|t|}{\log(|t|+2)}\Big)$$

with some constant $c_5 > 0$*.*

(2) *There are distribution functions* F_P*,* $P \geq 2$*, and* F *with the properties:*
- *The characteristic function of* F_P *is* Ψ_P*.*
- *The characteristic function of* F *is* Ψ*.*
- *The sequence* $(F_{P,x})_{x\geq 1}$ *converges weakly to* F_P*.*
- *The sequence* $(F_P)_{P\geq 2}$ *converges weakly to* F*.*
- F *is infinitely differentiable and all its derivatives are bounded.*

Proof. (1) The uniform convergence of $\Psi(t)$ on bounded sets follows from Lemma 3.5(2). The prime number theorem in arithmetic progressions gives

$$\sum_{p>x:\, p\equiv 1 \mod p} \frac{1}{p^2} \geq \frac{c_6}{x\log x}$$

as $x \to \infty$. From the definition it follows immediately that $|\psi(p,t)| \leq 1$ for all p,t. Now Lemma 3.5(1) gives for $|t| \geq c$

$$|\Psi(t)| \leq \prod_{p\geq c(|t|+1):\, p\equiv 1 \mod l} \exp\Big(-\frac{l-1}{4p^2}t^2\Big)$$
$$= \exp\Big(-\frac{t^2(l-1)}{4}\sum_{p\geq c(|t|+1):\, p\equiv 1 \mod l}\frac{1}{p^2}\Big)$$
$$\leq \exp\Big(-\frac{c_6(l-1)t^2}{4c(|t|+1)\log(c(|t|+1))}\Big).$$

(2) From Lemma 3.3 and Kolmogorov's Continuity Theorem (see, e.g., [7], Lemma 1.11) it follows that Ψ_P is the characteristic function of a distribution function F_P which is the weak limit of the sequence $(F_{P,x})_{x\geq 1}$. From part (1) it follows that $\lim_{P\to\infty}\Psi_P(t) = \Psi(t)$ uniformly for bounded t. The same argument as above now shows that Ψ is the characteristic function of a distribution function F such that $\lim_{P\to\infty} F_P = F$ weakly. The Fourier Inversion Theorem (see, e.g., [7], Lemma 1.10) and the fast decay of Ψ show that

$$F(z_2) - F(z_1) = \int_{-\infty}^{\infty} \frac{e^{-itz_2} - e^{-itz_1}}{it}\,\Psi(t)\,dt \tag{3.6}$$

for $z_1, z_2 \in \mathbb{R}$ at which F is continuous. In particular,

$$|F(z_2) - F(z_1)| \le \int_{-\infty}^{\infty} \Big| \frac{e^{-itz_2} - e^{-itz_1}}{it(z_2 - z_1)} \Big| |\Psi(t)| \, dt \cdot |z_2 - z_1| \ll |z_2 - z_1|$$

for these z_1, z_2. This shows that F is continuous everywhere and (3.6) holds for all z_1, z_2. The theorem on differentiation of parameter integrals now shows that $F \in C^\infty(\mathbb{R})$ and all its derivatives are bounded. □

4. Almost periodicity of the function r

Let $\mathcal{D}$ be the $\mathbb{C}$–linear space of periodic arithmetical functions and, for $q \ge 1$, let $\mathcal{D}^q$ be the closure of $\mathcal{D}$ with respect to $\|\cdot\|_q$ in the space of all functions with finite q–seminorm. Then $\mathcal{D}^q$ is a Banach space (see [18], Chapter VI, Theorem 1.4).

For $\Re s > 1$ and $m \in S_l$, we have

$$\zeta_{k_m}(s)\zeta(s)^{-1} = r_1(m,s)\, r_2(m,s)\, r_3(m,s),$$

where

$$r_1(m,s) := \prod_p \prod_{\mathfrak{p}|p:\, f(\mathfrak{p}/p)\ge 2} \Big(1 - \frac{1}{p^{f(\mathfrak{p}/p)s}}\Big)^{-1}, \quad r_2(m,s) := \Big(1 - \frac{1}{l^s}\Big)^{-\chi(m,p)},$$

$$r_3(m,s) := \prod_{p \ne l} \Big(1 - \frac{1}{p^s}\Big)^{-\chi(m,p)}.$$

Thus $r = r_1\, r_2\, r_3$, where

$$r_1 = \prod_p \gamma_{p2}, \quad r_2 = \gamma_{l1}, \quad r_3(m) = \left\{ \begin{array}{ll} r_3(m,1), & m \in S_l, \\ 0, & m \in \mathbb{N} \setminus S_l. \end{array} \right\}$$

Lemma 4.1. (1) *For every $\epsilon > 0$, the product $r_1(m,s)$ converges uniformly with respect to $m \in S_l$, $\Re s \ge 1/2 + \epsilon$.*

(2) *For $m \in S_l$, the function $r_1(m,\cdot)$ is holomorphic and zero–free on $\Re s > 1/2$; it is bounded on every half–plane $\Re s \ge 1/2 + \epsilon$ where $\epsilon > 0$.*

(3) *For every $q \ge 1$, we have $r_1, r_2 \in \mathcal{D}^q$.*

Proof. (1) follows easily from the condition $f(\mathfrak{p}/p) \ge 2$ in the product. (2) follows from this and the fact that the factors have no zeros in $\Re s > 1/2$.

(3) First we show that for p a rational prime and $q \ge 1$, we have $\gamma_{p1}, \gamma_{p2} \in \mathcal{D}^q$. From Lemma 3.1 it follows that for $m', m'' \in S_l$ with $m' \equiv m'' \mod p^{(l-1)^2+l+l}$, we have $\gamma_{pi}(m') = \gamma_{pi}(m'')$, $i = 1,2$. For a set $X \subseteq \mathbb{N}$, denote its indicator function by I_X. We see that there are $p^{(l-1)^2+l+1}$–periodic functions $\tilde{\gamma}_{pi}$ with $\gamma_{pi} = \tilde{\gamma}_{pi} I_{S_l}$. Since $I_{S_l} \in \mathcal{D}^q$ (see [18], Chapter VII, Theorem 4.1), we have $\gamma_{pi} \in \mathcal{D}^q$. In particular, $r_2 = \gamma_{l1} \in \mathcal{D}^q$.

Now let $q \ge 1$ and $\epsilon > 0$. From (1) it follows that there is some $P \ge 2$ such that $|r_1(m) - \prod_{p \le P} \gamma_{p2}(m)| \le \epsilon$ for all $m \in S_l$. Trivially this also holds for $m \in \mathbb{N} \setminus S_l$. From the above we know that $\gamma_{p2} \in \mathcal{D}^{q\pi(P)}$ for all $p \le P$ where $\pi(P)$ is the number of primes $\le P$. Thus $\prod_{p \le P} \gamma_{p2} \in \mathcal{D}^q$. So r_1 is the uniform limit of functions in $\mathcal{D}^q$ and therefore $r_1 \in \mathcal{D}^q$. □

Lemma 4.2. (1) *For $m \in S_l$, the function $r_3(m,\cdot)$ is holomorphic on $\Re s > 1/2$.*

(2) *There is some* $0 < \rho < 1$ *such that for all sufficiently small* $\epsilon > 0$, *there are constants* $c(\epsilon) > 0$ *and* $1/2 < \sigma_2(\epsilon) < 1$ *with the property: For all* $x \geq 1$ *and all* $m \in S_l$, $m \leq x$, *with the exception of* $O(x^{1-\rho})$ *many, we have*

$$r_3(m, s) \ll_\epsilon \exp(c(\epsilon)(\log x)^\epsilon)$$

for $\Re s = \sigma_2(\epsilon)$, $|\Im s| \leq (\log x)^2/4$, *and*

$$r_3(m, s) \ll (d_{k_m/\mathbb{Q}}(|\Im s| + 1))^{c_8}$$

for $\Re s = \sigma_2(\epsilon)$.

Proof. Since $\zeta_{k_m}(s)\zeta(s)^{-1}$ is entire (see [22] or [23]), part (1) follows from Lemma 4.1(2).

Part (2) is proved in the usual way. Let $1/2 < \sigma_0 < \sigma_1 < 1$ be fixed and $m \in S_l$, $m \leq x$, such that $\zeta_{k_m}(s)\zeta(s)^{-1}$ has no zeros in $\sigma_0 \leq \Re s \leq 1$, $|\Im s| \leq \log^2 x$. For $\Re s = 3/2$, we have

$$|\zeta_{k_m}(s)| \leq \zeta\Big(\frac{3}{2}\Big)^l \ll 1, \quad |\zeta(s)^{-1}| \leq \prod_p \Big(1 + \frac{1}{p^{3/2}}\Big) \ll 1.$$

The functional equation shows that

$$|\zeta_{k_m}(1-s)| \ll (d_{k_m/\mathbb{Q}}(|\Im s| + 1))^{c_6}, \quad |\zeta(1-s)^{-1}| \ll (|\Im s| + 1)^{c_6}$$

with some constant $c_6 > 0$. Furthermore, ζ_{k_m} grows polynomially in the strip $-1 \leq \Re s \leq 3/2$. The same holds for $\zeta(s)^{-1}$ if s is not too close to the zeros of $\zeta(s)$ (see [5], Chapter 17). Thus the Phragmén-Lindelöf Principle shows that

$$|r_3(m, s)| \ll (d_{k_m/\mathbb{Q}}(|\Im s| + 1))^{2c_6}$$

for $\sigma_0 \leq \Re s \leq 3/2$.

On $\Re s > 1$, the function

$$\log r_3(m, s) := \sum_{p \neq l} \sum_{k \geq 1} \frac{(-1)^k}{k\, p^{ks}}\, \chi(m, p)$$

is a logarithm of $r_3(m, s)$. Because of the assumption, it can be extended to the region $\sigma_0 \leq \Re s \leq 1$, $|\Im s| \leq \log^2 x$. Furthermore,

$$|\log r_3(m, s)| \ll 1 + \sum_p \frac{1}{p^{\Re s}}$$

for $\Re s > 1$. The Borel-Caratheodory theorem now gives

$$|\log r_3(m, s)| \ll \log d_{k_m/\mathbb{Q}} + \log\log x \ll \log x$$

for $\sigma_1 \leq \Re s \leq 3/2$, $|\Im s| \leq (\log x)^2/2$, since $d_{k_m/\mathbb{Q}} | l^l m^{l-1}$. From this it follows by Hadamard's Three Circles Theorem that

$$|\log r_3(m, s)| \ll_\epsilon (\log x)^\epsilon$$

for $x \gg_\epsilon 1$, $\Re s = 1 - \epsilon(1 - \sigma_1)/3$, $|\Im s| \leq (\log x)^2/4$.

It remains to show that the number

$$A(\sigma_0, x) := \#\big\{m \in S_l \,\big|\, m \leq x,\ \zeta_{k_m}(s)\zeta(s)^{-1} \text{ has a zero in } \sigma_0 \leq \Re s \leq 1,\ |\Im s| \leq \log^2 x\big\}$$

is negligible if σ_0 is choosen appropriately. By Kawada's density theorem [10], we see that

$$\sum_{m\in S_l:\, N<m\le 2N} N(m;1-\eta,T) \ll (NT)^{1-\eta}, \quad 1\le T\le N,$$

where $\eta = 1/(2000l^2)$ and $N(m;1-\eta,T)$ is the number of zeros of $\zeta_{k_m}(s)\zeta(s)^{-1}$ in $1-\eta \le \Re s \le 1$, $|\Im s| \le T$. Now put $N = 2^i$, $T = \log^2 x$, and sum over $2\log\log x/\log 2 < i \le \log x/\log 2$. This gives

$$A(1-\eta,x) \ll 2^{[2\log\log x/\log 2]+1} + \sum_{0\le i\le \log x/\log 2} (2^i\log^2 x)^{1-\eta} \ll (x\log^2 x)^{1-\eta}.$$

Choosing $0<\rho<\eta$, $\sigma_0 = 1-\eta$, $\sigma_0<\sigma_1<1$ and $\sigma_2(\epsilon) = 1-\epsilon(1-\sigma_1)/3$ finishes the proof. $\square$

The following lemma ist the main analytic tool to approximate r by periodic functions. For $m\in S_l$, let $\tilde\chi(m,\cdot):\mathbb{N}\to\mathbb{C}$ be multiplicative with

$$\tilde\chi(m,p^a) := \left\{ \begin{array}{ll} (-1)^a\binom{-\chi(m,p)}{a}, & p\ne l, \\ 0, & p=l. \end{array} \right\}$$

Lemma 4.3. *For every $\alpha>0$, there is a constant $c_9(\alpha)>0$ with the property: For $x\ge 1$, $N:=x^\alpha$, and all $m\in S_l$, $m\le x$, with the exception of $O(x^{1-\rho})$ many, we have*

$$r_3(m) = \sum_{n\ge 1}\frac{\tilde\chi(m,n)}{n}\, e^{-n/N} + O(x^{-c_9(\alpha)}).$$

Proof. For $m\in S_l$, p a rational prime and $a\ge 1$,

$$\begin{aligned} |\tilde\chi(m,p^a)| &\le \frac{(l-1)l(l+1)\cdots(l+a-2)}{a!} = \binom{l+a-2}{a} = \frac{(l+a-2)\cdots(a+1)}{(l-2)!} \\ &= \left(\frac{a}{l-2}+1\right)\left(\frac{a}{l-3}+1\right)\cdots\left(\frac{a}{1}+1\right) \le (a+1)^{l-2} = d(p^a)^{l-2}. \end{aligned}$$

Thus for $n\in\mathbb{N}$, we have

$$|\tilde\chi(m,n)| \le d(n)^{l-2}. \tag{4.1}$$

Now the Binomial series shows that for $\Re s>1$, we have

$$r_3(m,s) = \sum_{n\ge 1}\frac{\tilde\chi(m,n)}{n^s}, \tag{4.2}$$

where the Dirichlet series is absolutely convergent.

Choose $\epsilon>0$ so small that Lemma 4.2 can be applied. Let $x\ge 1$, $N:=x^\alpha$, and $m\in S_l$, $m\le x$, a non–exceptional number. Since $r_3(m,\cdot)$ grows polynomially in the strip $\sigma_2(\epsilon)\le\Re s\le 2$, the residue theorem gives

$$\begin{aligned} r_3(m) = r_3(m,1) =& \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} r_3(m,s)\,\Gamma(s-1)\,N^{s-1}ds \\ &- \int_{\sigma_2(\epsilon)-i\infty}^{\sigma_2(\epsilon)+i\infty} r_3(m,s)\,\Gamma(s-1)\,N^{s-1}ds. \end{aligned} \tag{4.3}$$

Since

$$\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\Gamma(s)\,y^{-s}ds=e^{-y},\quad \beta,y>0,$$

plugging (4.2) in the first term of (4.3) gives the value

$$\sum_{n\geq 1}\frac{\tilde\chi(m,n)}{n}\,e^{-n/N}.$$

Furthermore,

$$\Gamma(s-1)\ll(|\Im s|+1)^{c_7}e^{-\pi|\Im s|/2},\quad \Re s=\sigma_2(\epsilon),$$

with some constant $c_7>0$. Therefore Lemma 4.2 gives for the second term in (4.3) the estimate

$$\begin{aligned}&\int_{|t|\leq(\log x)^2/4}e^{c(\epsilon)(\log x)^\epsilon}(|t|+1)^{c_7}e^{-\pi|t|/2}N^{\sigma_2(\epsilon)-1}dt\\&\qquad+\int_{|t|\geq(\log x)^2/4}(x^{l-1}(|t|+1))^{c_8}(|t|+1)^{c_7}e^{-\pi|t|/2}N^{\sigma_2(\epsilon)-1}dt\\&\ll N^{\sigma_2(\epsilon)-1}\Big(e^{c(\epsilon)(\log x)^\epsilon}+x^{(l-1)c_8}e^{-(\pi/2-\epsilon)(\log x)^2/4}\Big)\ll x^{-c_9(\alpha)}\end{aligned}$$

with some constant $c_9(\alpha)>0$. □

Lemma 4.4. *For all $m\in S_l$, we have $r_3(m)\ll(\log m)^{l-1}$.*

Proof. From [14], Corollary 4 to Theorem 7.1, it follows that

$$r(m)\ll(\log d_{k_m/\mathbb{Q}})^{l-1}\ll(\log m)^{l-1}.$$

Since $r_1(m),r_2(m)\gg 1$, the statement follows. □

Proposition 4.5. *For all $q\geq 1$, we have $r_3\in\mathcal{D}^q$. For $q\geq 1$, we have $\|r_3\|_q^q\ll\exp(c_{12}\,q\log\log(q+2))$.*

Proof. Let $q\in\mathbb{N}$ and $\alpha>0$ be fixed. From Lemmas 4.3 and 4.4 and (4.1) it follows that for $L\geq 1$, $x\geq L^{1/\alpha}$ and $N:=x^\alpha$, we have

$$\begin{aligned}&\sum_{m\in S_l:\,m\leq x}\Big|r_3(m)-\sum_{1\leq n\leq L}\frac{\tilde\chi(m,n)}{n}\Big|^{2q}\\&\qquad\ll_q x^{1-\rho}\Big((\log x)^{2q(l-1)}+\Big(\sum_{1\leq n\leq L}\frac{d(n)^{l-2}}{n}\Big)^{2q}\Big)\\&\qquad\quad+\sum_{m\in S_l:\,m\leq x\text{ not an exception}}\Big(\Big|\sum_{1\leq n\leq L}\frac{\tilde\chi(m,n)}{n}(e^{-n/N}-1)\Big|^{2q}\\&\qquad\qquad+\Big|\sum_{n>L}\frac{\tilde\chi(m,n)}{n}\,e^{-n/N}\Big|^{2q}+x^{-2qc_9(\alpha)}\Big).\end{aligned}$$

Since $x\geq L^{1/\alpha}$, the first sum on the right hand side is

$$\ll\sum_{1\leq m\leq x}\Big(\sum_{1\leq n\leq L}\frac{d(n)^{l-2}}{n}\cdot\frac{n}{x^\alpha}\Big)^{2q}.$$

Thus

$$\limsup_{x\to\infty} \frac{1}{x} \sum_{m\in S_l:\, m\le x} \Big| r_3(m) - \sum_{1\le n\le L} \frac{\tilde\chi(m,n)}{n} \Big|^{2q}$$
$$\ll_q \limsup_{x\to\infty} \frac{1}{x} \sum_{m\in S_l:\, m\le x} \Big| \sum_{n>L} \frac{\tilde\chi(m,n)}{n}\, e^{-n/x^\alpha} \Big|^{2q}. \tag{4.4}$$

From Proposition 2.4 it follows that for all rational primes $p \neq l$, the function $\chi(\cdot,p)$ is p–periodic on S_l. Thus for all $n \in \mathbb{N}$, the function $\tilde\chi(\cdot,n)$ is n–periodic on S_l. Therefore it can be extended to an n–periodic function on $\mathbb{N}$ — which is again denoted by $\tilde\chi(\cdot,n)$ — such that for all $m \in \mathbb{N}$, the function $\tilde\chi(m,\cdot)$ is multiplicative. With this extension, we have

$$S_1 := \sum_{m\in S_l:\, m\le x} \Big| \sum_{n>L} \frac{\tilde\chi(m,n)}{n}\, e^{-n/N} \Big|^{2q} \le \sum_{1\le m\le x} \Big| \quad \Big|^{2q}$$
$$= \sum_{n_1,\dots n_{2q}>L} \frac{1}{n_1\dots n_{2q}}\, e^{-(n_1+\cdots n_{2q})/N}\, S_2(n_1,\dots,n_{2q}),$$

where

$$S_2(n_1,\dots,n_{2q}) := \sum_{1\le m\le x} \tilde\chi(m,n_1)\cdots\tilde\chi(m,n_{2q}).$$

Now we must estimate S_2. Assume first that there is a rational prime p with $p\|n_1\cdots n_{2q}$. By the Chinese Remainder Theorem,

$$\sum_{m \bmod n_1\cdots n_{2q}} \tilde\chi(m,n_1)\cdots\tilde\chi(m,n_{2q}) = \prod_{\tilde p^a\|n_1\cdots n_{2q}} \Big(\sum_{m \bmod \tilde p^a} \tilde\chi(m,\tilde p^{\operatorname{ord}_{\tilde p} n_1})\cdots\tilde\chi(m,\tilde p^{\operatorname{ord}_{\tilde p} n_{2q}}) \Big),$$

where $\tilde p$ runs through rational primes. The factor for $\tilde p = p$ is

$$\sum_{m \bmod p} \tilde\chi(m,p) = 0;$$

for $p \neq l$ this follows from Proposition 2.4, and for $p = l$ it is trivial. Thus

$$|S_2(n_1,\dots,n_{2q})| \le \sum_{m \bmod n_1\cdots n_{2q}} |\tilde\chi(m,n_1)\cdots\tilde\chi(m,n_{2q})| \le n_1\cdots n_{2q}(d(n_1)\cdots d(n_{2q}))^{l-2}.$$

If, on the other hand, $p^2|n_1\cdots n_{2q}$ for all prime divisors p of $n_1\cdots n_{2q}$, all we can say is that

$$|S_2(n_1,\cdots,n_{2q})| \le x\,(d(n_1)\cdots d(n_{2q}))^{l-2}.$$

With this we get the estimate

$$S_1 \ll_{\epsilon,q} \sum_{n_1,\dots,n_{2q}>L:\, n_1\cdots n_{2q} \text{ is squarefull}} \frac{1}{n_1\cdots n_{2q}}\, x\,(n_1\cdots n_{2q})^\epsilon$$
$$+ \sum_{n_1,\dots,n_{2q}>L:\, n_1\cdots n_{2q} \text{ is not squarefull}} \frac{1}{n_1\cdots n_{2q}}\, e^{-(n_1+\cdots n_{2q})/N}\,(n_1\cdots n_{2q})^{1+\epsilon}$$

$$\ll_{\epsilon,q} x \sum_{n>L^{2q}:\, n \text{ is squarefull}} \frac{d_{2q}(n)}{n^{1-\epsilon}} + \Big(\sum_{n\geq 0} n^{\epsilon} e^{-n/N}\Big)^{2q}.$$

Since $n^{\epsilon}\, e^{-n/N} \leq N^{\epsilon}(n/N)^{\epsilon}\, e^{-n/N} \ll_{\epsilon} N^{\epsilon}\, e^{-n/(2N)}$, the second term on the right hand side is

$$\ll N^{2q\epsilon}\big(1-e^{-1/(2N)}\big)^{-2q} \ll N^{2q(1+\epsilon)}.$$

Since

$$\sum_{n\geq 1 \text{ is squarefull}} \frac{1}{n^{1-2\epsilon}} = \prod_p \Big(1+\sum_{i\geq 2}\frac{1}{p^{(1-2\epsilon)i}}\Big)$$

converges, we have

$$S_1 \ll x\, s(L) + x^{2q(1+\epsilon)\alpha},$$

where $s(L)\to 0$ as $L\to\infty$. Thus it follows from (4.4) that

$$\Big\| r_3 - I_{S_l}\cdot \sum_{1\leq n\leq L} \frac{\tilde{\chi}(\cdot,n)}{n}\Big\|_{2q}^{2q} \ll_q s(L)$$

if we choose $0<\alpha<1/(2q(1+\epsilon))$. Since $I_{S_l}\in\mathcal{D}^{2q}$ and the sum above is periodic, we have $r_3\in\mathcal{D}^{2q}$. This holds for all $q\in\mathbb{N}$. Thus $r_3\in\mathcal{D}^{q}$ for all $q\geq 1$.

In particular, for $q\in\mathbb{N}$ we have

$$\|r_3\|_{2q}^{2q} = \lim_{L\to\infty} \Big\| I_{S_l}\cdot \sum_{1\leq n\leq L}\frac{\tilde{\chi}(\cdot,n)}{n}\Big\|_{2q}^{2q}.$$

Since $|\tilde{\chi}(m,p^a)| \leq \binom{l+a-2}{a}$, an argument as above gives

$$\begin{aligned}\sum_{m\in S_l:\, m\leq x}\Big|\sum_{1\leq n\leq L}\frac{\tilde{\chi}(m,n)}{n}\Big|^{2q} &\leq \sum_{1\leq n_1,\ldots,n_{2q}\leq L}\frac{1}{n_1\cdots n_{2q}} S_2(n_1,\ldots,n_{2q})\\ &\ll\; x \sum_{n\geq 1 \text{ squarefull}} \frac{1}{n}\sum_{n_1\cdots n_{2q}=n} \tilde{d}(n_1)\cdots\tilde{d}(n_{2q}) + C(L,q),\end{aligned}$$

where $\tilde{d}$ is multiplicative with $\tilde{d}(p^a)=\binom{l+a-2}{a}$, the $\ll$-constant is independent of q and $C(L,q)$ does not depend on x. Therefore,

$$\begin{aligned}\|r_3\|_{2q}^{2q} &\ll \sum_{n\geq 1 \text{ squarefull}}\frac{1}{n}\sum_{n_1\cdots n_{2q}=n}\tilde{d}(n_1)\cdots\tilde{d}(n_{2q})\\ &= \prod_p\Bigg(1+\sum_{a\geq 2}\frac{1}{p^a}\sum_{a_1,\ldots,a_{2q}\geq 0:\, a_1+\cdots a_{2q}=a}\ \prod_{i=1}^{2q}\binom{l+a_i-2}{a_i}\Bigg). \qquad (4.5)\end{aligned}$$

For $|z|<1$, we have $\sum_{a\geq 0}\binom{l+a-2}{a} z^a = (1-z)^{-(l-1)}$ and thus

$$\sum_{a\geq 0} z^a \sum_{a_1,\ldots,a_{2q}\geq 0:\, a_1+\cdots+a_{2q}=a}\ \prod_{i=1}^{2q}\binom{l+a_i-2}{a_i} = (1-z)^{-K}, \quad K:=2q(l-1).$$

Denote the p–th factor in (4.5) by $A(p)$. For $p \le K$, we have $A(p) \le (1-1/p)^{-K}$ and thus

$$\log A(p) \le \frac{K}{p-1} \ll \frac{K}{p}.$$

For $p > K$, we have

$$\begin{aligned} A(p) &= \Big(1-\frac{1}{p}\Big)^{-K} - \frac{K}{p} = \Big(1-\frac{1}{p^2}\Big)^{-K}\Big(\Big(1+\frac{1}{p}\Big)^{K} - \frac{K}{p}\Big(1-\frac{1}{p^2}\Big)^{K}\Big) \\ &= \Big(1-\frac{1}{p^2}\Big)^{-K}\Big(1+\sum_{\kappa=2}^{K}\binom{K}{\kappa}\frac{1}{p^\kappa} - \frac{K}{p}\sum_{\kappa=1}^{K}\binom{K}{\kappa}\Big(-\frac{1}{p^2}\Big)^{\kappa}\Big) \end{aligned}$$

and thus

$$\log A(p) \le \frac{K}{p^2-1} + \sum_{\kappa=2}^{K}\binom{K}{\kappa}\frac{1}{p^\kappa} - \frac{K}{p}\sum_{\kappa=1}^{K}\binom{K}{\kappa}\Big(-\frac{1}{p^2}\Big)^{\kappa}.$$

Since $p > K$, we have

$$\binom{K}{\kappa}\frac{1}{p^\kappa} \le \frac{1}{\kappa!}\Big(\frac{K}{p}\Big)^{\kappa} \le \frac{1}{\kappa!}\Big(\frac{K}{p}\Big)^{2}, \quad 2 \le \kappa \le K,$$

and

$$\frac{K}{p}\binom{K}{\kappa}\frac{1}{p^{2\kappa}} \le \frac{1}{\kappa!}\Big(\frac{K}{p}\Big)^{\kappa+1} \le \frac{1}{\kappa!}\Big(\frac{K}{p}\Big)^{2}, \quad 1 \le \kappa \le K.$$

Therefore $\log A(p) \ll K^2/p^2$. Plugging these estimates in (4.5) gives

$$\|r_3\|_{2q}^{2q} \ll \exp\Big(c_{10}\Big(\sum_{p\le K}\frac{K}{p} + \sum_{p>K}\frac{K^2}{p^2}\Big)\Big) \le \exp(c_{11}K\log\log K).$$

Since $\|r_3\|_t$ is non–decreasing in $t \ge 1$, the second statement follows. □

Now we can prove part of the main theorem.

Theorem 4.6. *For $q \ge 1$, we have $r \in \mathcal{D}^q$. The function r, restricted to S_l, has a limit distribution. For $q \ge 1$, the q–th moment of r on S_l exists and equals the q–th moment of its limit distribution.*

Proof. From Lemma 4.1(3) and Proposition 4.5 it follows that $r = r_1 r_2 r_3 \in \mathcal{D}^q$ for all $q \ge 1$. In particular, the q–th moment of r on S_l

$$\beta_q := \lim_{x\to\infty}\frac{1}{S_l(x)}\sum_{m\in S_l\colon m\le x} r(m)^q = \frac{1}{d(S_l)}\|r\|_q^q$$

exists. Since r_1 and r_2 are bounded, it follows from Proposition 4.5 that

$$\beta_q \ll (\sup r_1 r_2)^q\,\|r_3\|_q^q \ll \exp(c_{13}\,q\log\log(q+2)),$$

and in particular the power series

$$\Phi(z) := 1 + \sum_{q\ge1}\frac{\beta_q}{q!}z^q$$

has infinite radius of convergence. Since β_q is the limit of the q–th moments of the distribution functions $F_x(\log z)$, $z \in \mathbb{R}^+$, where

$$F_x(z) := \frac{1}{S_l(x)} \; \#\big\{m \in S_l \,\big|\, m \le x,\, r(m) \le e^z\big\}, \quad z \in \mathbb{R},$$

if follows from a theorem of Fréchet and Shohat (see, e.g., [7], Lemmas 1.43 and 1.44) that the sequence $(F_x(z))_{x\ge 1}$ converges weakly to a distribution function $F^*(z)$, and that

$$\beta_q = \int_{\mathbb{R}^+} z^q \, dF^*(\log z), \quad q \in \mathbb{N}.$$

It remains to show that $F^* \in C^\infty(\mathbb{R})$ and to compute the Euler product representation of $\Psi(t)$.

□

5. Smoothness of the limit distribution

In the proof of Theorem 4.6, the function r was approximated by periodic functions. This was useful to show almost periodicity and give estimates of the moments. Now r will be approximated by R_P which will be used to show that $F = F^*$.

Proposition 5.1. *We have* $\lim_{P\to\infty} \|r - R_p\|_2 = 0$.

Proof. For $L \ge 1$, define

$$s_1(L) := \Big\| r_3 - I_{S_l} \cdot \sum_{1\le n\le L} \frac{\tilde{\chi}(\cdot, n)}{n} \Big\|_2^2.$$

From the proof of Proposition 4.5 it follows that $s_1(L) \to 0$ as $L \to \infty$. For $P \ge 2$, define

$$s_2(P) := \sup_{m\in S_l} \Big| \prod_{p>P} \gamma_{p2}(m) - 1 \Big|.$$

From Lemma 4.1 it follows that $s_2(P) \to 0$ as $P \to \infty$. Furthermore,

$$s_3 := \sup_{m\in S_l,\, P\ge 2} \Big| r_2(m) \prod_{p\le P} \gamma_{p2}(m) \Big| < \infty.$$

Since $\tilde{\chi}(\cdot, n)$ is n–periodic, we have

$$s_4(L) := \sup_{m\in S_l} \Big| \sum_{1\le n\le L} \frac{\tilde{\chi}(m, n)}{n} \Big| < \infty.$$

Thus for $m \in S_l$, $P \ge L \ge l$, we have

$$\Big| r_1(m)\, r_2(m) \sum_{1\le n\le L} \frac{\tilde{\chi}(m, n)}{n} - R_P(m) \Big|$$

$$= \Big| r_2(m) \prod_{p\le P} \gamma_{p2}(m) \Big| \cdot \Big| \Big(\prod_{p>P} \gamma_{p2}(m) - 1 \Big) \sum_{1\le n\le L} \frac{\tilde{\chi}(m, n)}{n} + \sum_{1\le n\le L} \frac{\tilde{\chi}(m, n)}{n} - \prod_{p\le P:\, p\ne l} \gamma_{p1}(m) \Big|$$

$$\le s_3 \Big(s_2(P)\, s_4(L) + \Big| \sum_{1\le n\le L} \frac{\tilde{\chi}(m, n)}{n} - \prod_{p\le P:\, p\ne l} \gamma_{p1}(m) \Big| \Big).$$

Since $r_1\, r_2$ is bounded, we get

$$\begin{aligned}\|r - R_P\|_2 &\le \Big\| r - I_{S_l} r_1\, r_2 \sum_{1\le n\le L} \frac{\tilde\chi(\cdot,n)}{n}\Big\|_2 + \Big\| I_{S_l} r_1\, r_2 \sum_{1\le n\le L} \frac{\tilde\chi(\cdot,n)}{n} - R_P\Big\|_2 \\ &\ll s_1(L)^{1/2} + s_2(P) s_4(L) + \Big\| I_{S_l} \sum_{1\le n\le L} \frac{\tilde\chi(\cdot,n)}{n} - I_{S_l} \prod_{p\le P\colon p\ne l} \gamma_{p1}\Big\|_2.\end{aligned}$$

For $M \ge L$, define $a_{P,L,M}(n) := -1$ if $M \ge n > L$ and $p \le P$ for all $p|n$. Define $a_{P,L,M}(n) := 0$ otherwise. Then for $x \ge 1$,

$$\begin{aligned}&\sum_{m\in S_l\colon m\le x} \Big| \sum_{1\le n\le L} \frac{\tilde\chi(m,n)}{n} - \prod_{p\le P\colon p\ne l} \gamma_{p1}(m)\Big|^2 \\ &\qquad = \sum_{m\in S_l\colon m\le x} \Big| \sum_{n\ge 1} \frac{\tilde\chi(m,n) a_{P,L,M}(n)}{n} - \sum_{n>M\colon p|n\Rightarrow p\le P} \frac{\tilde\chi(m,n)}{n}\Big|^2 \\ &\qquad \ll \sum_{m\le x} \Big| \sum_{n\ge 1} \frac{\tilde\chi(m,n) a_{P,L,M}(n)}{n}\Big|^2 + x\Big(\sum_{n>M\colon p|n\Rightarrow p\le P} \frac{d(n)^{l-2}}{n}\Big)^2.\end{aligned}$$

Denote the second sum on the right hand side by $s_5(M,P)$. Since

$$\sum_{n\ge 1\colon p|n\Rightarrow p\le P} \frac{d(n)^{l-2}}{n} < \infty,$$

it follows that $s_5(M,P) \to 0$ as $M \to \infty$ for fixed $P \ge 2$. Since $a_{P,L,M}(n) = 0$ for $n > M$, an argument similar to that in the proof of Proposition 4.5 shows that

$$\begin{aligned}&\Big\| I_{S_l} \sum_{1\le n\le L} \frac{\tilde\chi(\cdot,n)}{n} - I_{S_l} \prod_{p\le P\colon p\ne l} \gamma_{p1}\Big\|_2^2 \\ &\ll \sum_{n_1,n_2\ge 1\colon n_1 n_2 \text{ squarefull}} \frac{(d(n_1)\, d(n_2))^{l-2}}{n_1\, n_2} \big| a_{P,L,M}(n_1) a_{P,L,M}(n_2)\big| + s_5(M,P)^2 \\ &\ll \sum_{n\ge L^2 \text{ squarefull}} \frac{1}{n^{2/3}} + s_5(M,P)^2.\end{aligned}$$

Thus for all $M \ge L$, $P \ge L \ge l$, we have

$$\|r - R_P\|_2 \ll s_1(L)^{1/2} + s_2(P)\, s_4(L) + \sum_{n\ge L^2 \text{ squarefull}} \frac{1}{n^{2/3}} + s_5(M,P)^2.$$

Letting $M \to \infty$ gives

$$\|r - R_P\|_2 \ll s_1(L)^{1/2} + s_2(P)\, s_4(L) + \sum_{n\ge L^2 \text{ squarefull}} \frac{1}{n^{2/3}}$$

for $P \ge L \ge l$. Given $\epsilon > 0$ there is an $L \ge l$ with $\sum_{n\ge L^2 \text{ squarefull}} n^{-2/3} \le \epsilon$ and $s_1(L) \le \epsilon^2$. Then there is a $P_0 \ge L$ such that for $P \ge P_0$, we have $s_2(P)\, s_4(L) \le \epsilon$ and thus $\|r - R_P\|_2 \ll \epsilon$. □

With Lemma 3.6 and Proposition 5.1 the following theorem can be proved as in [17], end of section 5.

Theorem 5.2. *On S_l, the function r has the limit distribution F.*

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Mathematisches Institut, Albert-Ludwigs-Universität, Eckerstr. 1, D-79104 Freiburg
E-mail address: manfred.peter@math.uni-freiburg.de